

BLOW UP FOR THE CRITICAL GKDV EQUATION I: DYNAMICS NEAR THE SOLITON

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ABSTRACT. We consider the mass critical (gKdV) equation $u_t + (u_{xx} + u^5)_x = 0$ for initial data in H^1 close to the soliton, which is a canonical *mass critical* problem. In the earlier works, [15, 24, 17], finite or infinite time blow up is proved for non positive energy solutions, and the solitary wave is shown to be the universal blow up profile. For well localized initial data, finite time blow up with an upper bound on blow up rate is obtained in [18].

In this paper, we fully revisit the analysis for (gKdV) in light of the recent progress made on the study of critical dispersive blow up problems [29, 35, 31, 30]. For a class of initial data close to the soliton, we show that three scenario only can occur: (i) the solution leaves any small neighborhood of the modulated family of solitary waves in the scale invariant L^2 norm; (ii) the solution is global and converges to a solitary wave as $t \rightarrow +\infty$; (iii) the solution blows up in finite time in a universal regime with speed:

$$\|u_x(t)\|_{L^2} \sim \frac{C(u_0)}{T-t}.$$

The regimes (i) and (iii) are moreover *stable*. We also show that nonpositive energy initial data yield finite time blow up, and obtain the classification of the solitary wave at zero energy as in [29].

1. Introduction

1.1. Setting of the problem. We consider the L^2 -critical generalized Korteweg-de Vries equation (gKdV)

$$\text{(gKdV)} \quad \begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (1.1)$$

The Cauchy problem is locally well posed in the energy space H^1 from Kenig, Ponce and Vega [10], and given $u_0 \in H^1$, there exists a unique¹ maximal solution $u(t)$ of (1.1) in $C([0, T), H^1)$ with either $T = +\infty$, or $T < +\infty$ and then $\lim_{t \rightarrow T} \|u_x(t)\|_{L^2} = +\infty$. The mass and the energy are conserved by the flow: $\forall t \in [0, T)$,

$$M(u(t)) = \int u^2(t) = M_0, \quad E(u(t)) = \frac{1}{2} \int u_x^2(t) - \frac{1}{6} \int u^6(t) = E_0,$$

where $M_0 = M(u_0)$, $E_0 = E(u_0)$, and the scaling symmetry ($\lambda > 0$)

$$u_\lambda(t, x) = \lambda^{\frac{1}{2}} u(\lambda^3 t, \lambda x)$$

leaves invariant the L^2 norm so that the problem is *mass critical*.

The family of travelling wave solutions

$$u(t, x) = \lambda_0^{-\frac{1}{2}} Q(\lambda_0^{-1}(x - \lambda_0^{-2}t - x_0)), \quad (\lambda_0, x_0) \in \mathbb{R}_+^* \times \mathbb{R},$$

¹in a certain sense

with

$$Q(x) = \left(\frac{3}{\cosh^2(2x)} \right)^{\frac{1}{4}}, \quad Q'' + Q^5 = Q, \quad E(Q) = 0, \quad (1.2)$$

plays a distinguished role in the analysis. From variational argument [41], H^1 initial data with subcritical mass $\|u_0\|_{L^2} < \|Q\|_{L^2}$ generate global and H^1 bounded solutions $T = +\infty$.

For $\|u_0\|_{L^2} \geq \|Q\|_{L^2}$, the existence of blow up solutions has been a long standing open problem. In particular, unlike for the analogous Schrödinger problem, there exists no simple obstruction to global existence. The study of singularity formation for small super critical mass H^1 initial data

$$\|Q\|_{L^2} \leq \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*, \quad \alpha^* \ll 1 \quad (1.3)$$

has been developed in a series of works by Martel and Merle [15, 16, 24, 17, 19, 18] where two new sets of tools are introduced:

- monotonicity formula and L^2 type localized virial identities to control the flow near the solitary wave;
 - rigidity Liouville type theorems to classify the asymptotic dynamics of the flow.
- In particular, the first proof of blow up in finite or infinite time is obtained for initial data

$$u_0 \in H^1 \text{ with (1.3) and } E(u_0) < 0. \quad (1.4)$$

The proof is indirect and based on a classification argument: the solitary wave is characterized as the unique universal attractor of the flow in the singular regime. If $u(t)$ blows up in finite or infinite time T with (1.3), then the flow admits near blow up time a decomposition

$$u(t, x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} (Q + \varepsilon) \left(t, \frac{x - x(t)}{\lambda(t)} \right) \quad \text{with } \varepsilon(t) \rightarrow 0 \text{ in } L^2_{\text{loc}} \text{ as } t \rightarrow T. \quad (1.5)$$

Then, in [18], for well localized initial data

$$u_0 \text{ satisfying (1.4) and } \int_{x' > x} u_0^2(x') dx' < \frac{C}{x^6} \text{ for } x > 0, \quad (1.6)$$

blow up is proved to occur in finite time T with an upper bound on a sequence $t_n \rightarrow T$:

$$\|u_x(t_n)\|_{L^2} \leq \frac{C(u_0)}{T - t_n}, \quad (1.7)$$

by a dynamical proof².

For the critical mass problem $\|u_0\|_{L^2} = \|Q\|_{L^2}$, assuming in addition the following decay $\int_{x' > x} u_0^2(x') dx' < \frac{C}{x^3}$ for $x > 0$, it was proved in [19] that the solution is global and does not blowup in infinite time.

1.2. Generic blow up for critical problems. In the continuation of these works, the program developed by Merle and Raphaël [25, 26, 27, 7, 34, 28, 29] for the mass critical nonlinear Schrödinger equation

$$(NLS) \quad \begin{cases} i\partial_t u + \Delta u + |u|^{\frac{4}{N}} u = 0, & (t, x) \in [0, T) \times \mathbb{R}^N. \\ u|_{t=0} = u_0 \end{cases} \quad (1.8)$$

in dimensions $1 \leq N \leq 5$ has led to a complete description of the stable blow up scenario near the solitary wave Q which is the unique H^1 nonnegative solution up to translation to $\Delta Q - Q + Q^{1+\frac{4}{N}} = 0$. This problem displays a similar structure like the critical (gKdV). Initial data in H^1 with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ are global and

²arguing directly on the solution itself.

bounded, [41]. For $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$, Merle [23] proved that the only blow up solution (up to the symmetries of the equation) is

$$S(t, x) = \frac{1}{t^{N/2}} e^{-i(\frac{|x|^2}{4t} - \frac{1}{t})} Q\left(\frac{x}{t}\right). \quad (1.9)$$

For small super critical mass H^1 initial data

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*, \quad \alpha^* \ll 1, \quad (1.10)$$

an H^1 open set of solutions is exhibited where solutions blow up in finite time at log-log speed:

$$\|\nabla u(t)\|_{L^2} \sim C^* \sqrt{\frac{\log|\log(T-t)|}{T-t}}. \quad (1.11)$$

Moreover, nonpositive energy solutions belong to this set of generic blow up. This double log correction to self similarity for stable blow up was conjectured from numerics by Landman, Papanicolou, Sulem and Sulem [38], and a family of such solutions was rigorously constructed by a different approach by Perelman in dimension $N = 1$, [39]. Blow up solutions of the type (1.9) ($\|u(t)\|_{H^1} \sim \frac{1}{t}$), constructed by Bourgain, Wang [1], see also Krieger, Schlag [13], correspond to an *unstable threshold dynamics* as proved in Merle, Raphaël, Szeftel [30]. Finally, under (1.10), the quantization of the focused mass at blow up is proved

$$|u(t)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=x(T)} + |u^*|^2, \quad u^* \in L^2. \quad (1.12)$$

More recently, natural connections have been made between mass critical problems and *energy critical* problems. For the energy critical wave map problem, after the pioneering work [40], a complete description of a generic finite time blow up dynamics (log correction to the self similar speed) was given by Raphaël, Rodnianski [35], while *unstable regimes* with different speeds were constructed by Krieger, Schlag, Tataru [14]. See also Merle, Raphael, Rodnianski [31] for the treatment of the Schrödinger map system and Raphaël, Schweyer [36] for the parabolic harmonic heat flow.

The general outcome of these works is twofold.

First the *sharp* derivation of the blow up speed in the *generic* regime relies on a detailed analysis of the structure of the solution near collapse, and takes in particular into account slowly decaying tails in the computation of the leading order blow up profile. These tails correspond to the leading order dispersive phenomenon which drives the speed of concentration and the rate of dispersion, both being intimately linked.

Second, a robust analytic approach has been developed in a nowadays more unified framework. In particular, the control of the solution in the singular regime relies on mixed energy/Morawetz or Virial type estimates adapted to the flow which have been used in various settings, see in particular [29], [37], [35], [31].

1.3. Statement of the results. The aim of the paper is to classify the gKdV dynamics for H^1 solutions close to the soliton and with decay on the right. In particular, we aim at recovering the more refined description of the flow obtained for the L^2 critical NLS equation.

More precisely, let us define the L^2 modulated tube around the soliton manifold:

$$\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 \text{ with } \inf_{\lambda_0 > 0, x_0 \in \mathbb{R}} \left\| u - \frac{1}{\lambda_0^{\frac{1}{2}}} Q\left(\frac{\cdot - x_0}{\lambda_0}\right) \right\|_{L^2} < \alpha^* \right\} \quad (1.13)$$

and consider the set of initial data

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 \text{ with } \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{y>0} y^{10} \varepsilon_0^2 < 1 \right\}.$$

Here α_0, α^* are universal constants with

$$0 < \alpha_0 \ll \alpha^* \ll 1. \quad (1.14)$$

Our aim is to classify the flow for data $u_0 \in \mathcal{A}$. First, we fully describe the blow up solutions in the tube \mathcal{T}_{α^*} : there is only one blow up type, which is stable. We then show that in fact only three scenario occur:

- stable blow up with $1/(T-t)$ speed;
- convergence to a solitary wave in large time;
- stable defocusing behavior (the solution leaves the tube \mathcal{T}_{α^*} in finite time).

We first claim:

Theorem 1.1 (Blow up near the soliton in \mathcal{A}). *There exist universal constants $0 < \alpha_0 \ll \alpha^* \ll 1$ such that the following holds. Let $u_0 \in \mathcal{A}$.*

- (i) Nonpositive energy blow up. *If $E(u_0) \leq 0$ and u_0 is not a soliton, then $u(t)$ blows up in finite time and, for all $t \in [0, T)$, $u(t) \in \mathcal{T}_{\alpha^*}$.*
- (ii) Description of blow up. *Assume that $u(t)$ blows up in finite time T and that for all $t \in [0, T)$, $u(t) \in \mathcal{T}_{\alpha^*}$. Then there exists $\ell_0 = \ell_0(u_0) > 0$ such that*

$$\|u_x(t)\|_{L^2} \sim \frac{\|Q'\|_{L^2}}{\ell_0(T-t)} \quad \text{as } t \rightarrow T. \quad (1.15)$$

Moreover, there exist $\lambda(t)$, $x(t)$ and $u^* \in H^1$, $u^* \neq 0$, such that

$$u(t, x) - \frac{1}{\lambda^{\frac{1}{2}}(t)} Q\left(\frac{x - x(t)}{\lambda(t)}\right) \rightarrow u^* \quad \text{in } L^2 \quad \text{as } t \rightarrow T, \quad (1.16)$$

where

$$\lambda(t) \sim \ell_0(T-t), \quad x(t) \sim \frac{1}{\ell_0^2(T-t)} \quad \text{as } t \rightarrow T, \quad (1.17)$$

$$\int_{x>R} (u^*)^2(x) dx \sim \frac{\|Q\|_{L^1}^2}{8\ell_0 R^2} \quad \text{as } R \rightarrow +\infty. \quad (1.18)$$

- (iii) Openness of the stable blow up. *Assume that $u(t)$ blows up in finite time T and that for all $t \in [0, T)$, $u(t) \in \mathcal{T}_{\alpha^*}$. Then there exists $\rho_0 = \rho_0(u_0) > 0$ such that for all $v_0 \in \mathcal{A}$ with $\|v_0 - u_0\|_{H^1} < \rho_0$, the corresponding solution $v(t)$ blows up in finite time $T(v_0)$ as in (ii).*

Comments on Theorem 1.1

1. *Blow up speed:* An important feature of Theorem 1.1 is the derivation of the stable blow up speed for $u_0 \in \mathcal{A}$:

$$\|u_x(t)\|_{L^2} \sim \frac{C}{T-t} \quad (1.19)$$

which implies that $x(t) \rightarrow +\infty$ as $t \rightarrow T$. Such a blow up rate confirms the conjecture formulated in [18] for $E_0 < 0$. Recall that for $u_0 \in \mathcal{A}$ and $E_0 < 0$, assuming some a priori global information on the \dot{H}^1 norm for all time in [18], one could deduce (1.19). The derivation of such a bound is the key to the proof of Theorem 1.1. This blow up speed is very far above the scaling law $\|u_x\|_{L^2} \sim 1/(T-t)^{\frac{1}{3}}$ (see [31], [36] for a similar phenomenon for energy critical geometrical problems).

2. *Structure of u^** : The decay of u^* in L^2 is directly related to the blow up speed $\frac{\|Q'\|_{L^2}}{\ell_0(T-t)}$, itself related to the speed of ejection of mass in time from the rescaled soliton, similarly like for the critical (NLS), see [28]. Note that the Cauchy problem is wellposed in L^2 , so that the L^2 convergence (1.16) is relevant. It is an open question but very likely that the convergence in (1.16) holds in H^1 since the left hand side is shown to be bounded in H^1 and u^* is in H^1 . The fact that $u^* \in H^1$ is in contrast with the stable regime for critical NLS, where the accumulation of ejected mass from the rescaled soliton implies that $u^* \notin L^p$, $p > 2$. Here we still observe some ejection of mass from the soliton, but since the concentration point $x(t)$ of the soliton is going to infinity, the mass does not accumulate at a fixed point and gives the tail of u^* . More generally, the regularity of u^* is directly connected to the blow up speed and the strength of deviation from self similarity, see [36], [31].

3. *On localization on the right*: Let us stress the importance of the decay assumption on the right in space for the initial data which was already essential in [18], [19]. Indeed, in contrast with the NLS equation, the universal dynamics can not be seen in H^1 since an additional assumption of decay to the right is required:

- In part II of this work [21], we construct a minimal mass blow up solution with $1/(T-t)$ blow up. The initial data is in H^1 and decays slowly on the right³. Thus, the blow up set without decay assumption on the right is *not open* in H^1 .

- For negative energy solutions with initial data with slow decay on the right (so that Theorem 1.1 and [18] do not apply), we expect the existence of solutions with different blow up speeds $1/(T-t)^\alpha$, $\alpha > 1$.

Note that there is however no sharpness in the y^{10} weight in Theorem 1.1.

4. *Dynamical characterization of Q* : Recall from the variational characterization of Q that $E(u_0) \leq 0$ implies $\|u_0\|_{L^2} > \|Q\|_{L^2}$, unless $u_0 \equiv Q$ up to scaling and translation symmetries. Theorem 1.1 therefore recovers the dynamical classification of Q as the unique global zero energy solution in \mathcal{A} like for the mass critical (NLS), see [29]. The proof of this type of result is delicate, and one needs to rule out a scenario of vanishing of the energy of the radiation specific to the zero energy case. Here, we expect this result to hold without decay assumption (no global H^1 energy zero solution close to Q exists except Q).

We now claim the following rigidity of the flow for data in \mathcal{A} :

Theorem 1.2 (Rigidity of the dynamics in \mathcal{A}). *There exist universal constants $0 < \alpha_0 \ll \alpha^* \ll 1$ such that the following holds. Let $u_0 \in \mathcal{A}$.*

Then, one of the following three scenarios occurs:

(Exit) *There exists $t^* \in (0, T)$ such that $u(t^*) \notin \mathcal{T}_{\alpha^*}$.*

(Blow up) *For all $t \in [0, T)$, $u(t) \in \mathcal{T}_{\alpha^*}$ and the solution blows up in finite time $T < +\infty$ in the regime described by Theorem 1.1.*

(Soliton) *The solution is global, for all $t \geq 0$, $u(t) \in \mathcal{T}_{\alpha^*}$, and there exist $\lambda_\infty > 0$, $x(t)$ such that*

$$\lambda_\infty^{\frac{1}{2}} u(t, \lambda_\infty \cdot + x(t)) \rightarrow Q \quad \text{in } H_{\text{loc}}^1 \text{ as } t \rightarrow +\infty, \quad (1.20)$$

$$|\lambda_\infty - 1| \leq o_{\alpha_0 \rightarrow 0}(1), \quad x(t) \sim \frac{t}{\lambda_\infty^2} \quad \text{as } t \rightarrow +\infty \quad (1.21)$$

³this is mandatory from [19]: no minimal mass blow up for data with decay on the right

Comments on Theorem 1.2

1. *Stable/unstable manifold:* All three possibilities are known to occur for an infinite set of initial data. Moreover, the sets of initial data leading to (Exit) and (Blow up) are both open in \mathcal{A} by perturbation of the data in H^1 . For $\int u_0^2 < \int Q^2$, only the (Exit) case can occur and for $E_0 < 0$, only (Blow up) can occur. From the proof of Theorem 1.2, the (Soliton) dynamics can be achieved as threshold dynamics between the two stable regimes (Exit) and (Blow up) as in [3], [8], [31]. More precisely, given $b \in \mathbb{R}$ small, let Q_b be the suitable perturbation of Q build in Lemma 2.4, and ε_0 be a suitable small perturbation satisfying the orthogonality conditions (2.20). Then there exists $b_0 = b(\varepsilon_0)$ such that the solution to (gKdV) with initial data $Q_{b_0} + \varepsilon_0$ satisfies (Soliton). The Lipschitz regularity of the flow $\varepsilon_0 \rightarrow b(\varepsilon_0)$ needed to build a smooth manifold remains to be proved, see [13] for related constructions. Note also that solutions that scatter to Q in the regime (Soliton) were constructed dynamically by Côte [2].

2. *Classification of the flow in \mathcal{A} .* Theorem 1.2 is a first step towards a complete classification of the flow for initial data in \mathcal{A} . Its structure is reminiscent from classification results obtained by Nakanishi and Schlag [32], [33] for super critical wave and Schrödinger equations. These results were proved using classification arguments based on the Kenig, Merle concentration compactness approach [11], the classification of critical dynamics by Duyckaerts, Merle [5], see also [6], and eventually a *no return lemma*. In the analogue of the (Exit) regime, this lemma shows that the solution cannot come back close to solitons and in fact scatters. In the critical situations, such an analysis is more delicate and incomplete, see [12], and both the blow statements and the no return lemma in [32], [33] rely on a specific algebraic structure - the virial identity - which does not exist for (gKdV). In the continuation of Theorem 1.2, what remains to be done to fully describe the flow for data $u_0 \in \mathcal{A}$ is to answer the question:

what happens after t^* in the (Exit) regime?

In [21], the second part of this work, we propose a new approach to answer this question related to the understanding of the threshold dynamics. We will proceed in two steps:

- (1) We prove the *existence and uniqueness in H^1* of a minimal mass blow up solution $\|u_0\|_{L^2} = \|Q\|_{L^2}$. From [19], this solution has slow decay to the right and is global on the left in time.
- (2) We then show that in the (Exit) case of Theorem 1.2, the solution is at time t^* L^2 close to the unique minimal mass blow up solution.

Having in mind the properties of threshold solutions for H^1 critical NLS and wave equations ([4, 5]), and the case of the L^2 critical NLS equation (the solution $S(t)$ in (1.9) scatters), it is natural to expect that the minimal mass blow up solution of (gKdV) also scatters in negative time. *Assuming this* and because scattering is open in the critical L^2 space, we obtain that (Exit) implies scattering. In other words, we prove in [21] that all solutions scatter in the (Exit) regime if and only if the unique H^1 minimal mass blow up solution scatters to the left. This ends the classification of the flow in \mathcal{A} , in particular the only blow-up regime is the $1/(T-t)$ universal blow-up regime of Theorem 1.1 and it is stable.

3. *Finite/Infinite dimensional dynamics.* The proof of Theorem 1.2 relies on a detailed description of the flow. We will show that before the (Exit) time t^* , the

solution admits a decomposition

$$u(t, x) = \frac{1}{\lambda^{\frac{1}{2}}(t)}(Q_{b(t)} + \varepsilon) \left(t, \frac{x - x(t)}{\lambda(t)} \right)$$

where Q_b is a suitable $O(b)$ deformation of the solitary wave profile, and there holds the bound

$$\|\varepsilon\|_{H_{\text{loc}}^1} \ll b.$$

We then extract the universal finite dimensional system which drives the geometrical parameters:

$$\frac{ds}{dt} = \frac{1}{\lambda^3}, \quad -\frac{\lambda_s}{\lambda} = b, \quad b_s + 2b^2 = 0. \quad (1.22)$$

It is easily seen that starting from $\lambda(0) = 1$, $b(0) = b_0$, the phase portrait of the dynamical system (1.22) is:

- (1) for $b_0 < 0$, $\lambda(t) = 1 + |b_0|t$, $t \geq 0$, stable;
- (2) for $b_0 = 0$, $\lambda(t) = 1$, $t \geq 0$, unstable;
- (3) for $b_0 > 0$, $\lambda(t) = b_0(T - t)$ with $T = \frac{1}{b_0}$, stable.

We may then reword Theorem 1.2 by saying that the infinite dimensional system (gKdV) for data $u_0 \in \mathcal{A}$ is governed to leading order by the universal finite dimensional dynamics (1.22). This is a non trivial claim due to the non linear structure of the problem, and the proof relies on a *rigidity* formula when measuring the interaction of the radiative term ε with the ODE's (1.22), see Lemma 4.3. Let us stress that the assumption of decay to the right is fundamental here, and we expect that slow decaying tails may force a different coupling with new leading order ODE's. Finally, note that like for the finite dimensional system (1.22), the three scenarios of Theorem 1.2 can be seen on $\lambda(t)$ only and are equivalently characterized by:

- (Soliton) for all t , $\lambda(t) \in [\frac{1}{2}, 2]$;
- (Exit) there exists $t_0 > 0$ such that $\lambda(t_0) > 2$;
- (Blow up) there exists $t_0 > 0$ such that $\lambda(t_0) < \frac{1}{2}$.

We expect that results such as Theorem 1.2 (classification of the dynamics close to the solitary waves) can be proved similarly for other problems like the NLS equation, the wave equation, etc.

Notation. Let the linearized operator close to Q be:

$$Lf = -f'' + f - 5Q^4f. \quad (1.23)$$

We introduce the generator of L^2 scaling:

$$\Lambda f = \frac{1}{2}f + yf'.$$

For a given generic small constant $0 < \alpha^* \ll 1$, $\delta(\alpha^*)$ denotes a generic small constant with

$$\delta(\alpha^*) \rightarrow 0 \quad \text{as} \quad \alpha^* \rightarrow 0.$$

We note the L^2 scalar product:

$$(f, g) = \int f(x)g(x)dx.$$

1.4. Strategy of the proof. We give in this section a brief insight into the proof of Theorems 1.1 and 1.2. As mentioned before, we are pushing further the dynamical analysis of the problem initiated in [18]. We will not use rigidity arguments as for the theory in H^1 (see [24], [17]). Nevertheless, we will use tools introduced to prove such rigidity arguments, such as modulation theory, L^2 and energy monotonicity, local Virial identities and weighted estimates for $x > 0$. However, the proofs here are self-contained, except for the Virial estimates, for which we refer to [15] and [17].

(i). *Formal derivation of the law*

We start as in [25], [29], [35] by refining the blow up profile and considering an approximation to the renormalized equation. We look for a solution to (gKdV) of the form

$$u(t, x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} Q_{b(s)} \left(\frac{x - x(t)}{\lambda(t)} \right), \quad \frac{ds}{dt} = \frac{1}{\lambda^3}, \quad \frac{x_s}{\lambda} = 1, \quad b = -\frac{\lambda_s}{\lambda}, \quad (1.24)$$

which leads to the slowly modulated self similar equation:

$$b_s \frac{\partial Q_b}{\partial b} + b \Lambda Q_b + (Q_b'' - Q_b + Q_b^5)' = 0. \quad (1.25)$$

A formal derivation of the generic blow up speed can be obtained as follows: look for a slowly modulated ansatz

$$Q_b = Q + bP + b^2 P_2 + \dots, \quad b_s = -c_2 b^2 + c_3 b^3 + \dots$$

where the unknowns are P and c_2, c_3 . Let the linearized operator close to Q be given by (1.23), then the order b expansion leads to the equation

$$(LP)' = \Lambda Q$$

which thanks to the critical orthogonality condition $(Q, \Lambda Q) = 0$ can be solved for a function P that decays exponentially to the right, but displays a non trivial tail on the left $\lim_{y \rightarrow -\infty} P(y) \neq 0$. At the level b^2 , a similar *flux type* computation⁴ reveals that the P_2 equation can be solved with a similar profile for the value $c_2 = 2$ only⁵. This corresponds to the formal dynamical system

$$-\frac{\lambda_s}{\lambda} = b, \quad b_s + 2b^2 = \lambda^2 \frac{d}{ds} \left(\frac{b}{\lambda^2} \right) = 0, \quad \frac{ds}{dt} = \frac{1}{\lambda^3} \quad (1.26)$$

which after reintegration yields finite time blow up for $b(0) > 0$ with

$$\lambda(t) = c(u_0)(T - t).$$

(ii). *Decomposition of the flow and modulation equations (section 2)*

For the analysis, it is enough to work with the localized approximate self similar profile

$$Q_b = Q + \chi(|b|^\gamma y) P(y)$$

for some well chosen⁶ $\gamma > 0$. As long as the solution remains in the tube \mathcal{T}_{α^*} , we may introduce the nonlinear decomposition of the flow:

$$u(t, x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} (Q_{b(t)} + \varepsilon) \left(t, \frac{x - x(t)}{\lambda(t)} \right), \quad (1.27)$$

⁴see (2.43)

⁵otherwise, P_2 grows exponentially on the right or the left.

⁶see Lemma 2.4, we can take $\gamma = \frac{3}{4}$

where the three time dependent parameters are adjusted to ensure suitable orthogonality conditions⁷ for ε . A specific feature of the (KdV) flow is that the generalized null space of the full linearized operator $(L)'$ close to Q involves *badly* localized functions in the right, and hence the modulations equations driving the parameters are roughly speaking of the form

$$\frac{\lambda_s}{\lambda} + b = \frac{dJ_1}{ds} + O(\|\varepsilon\|_{H_{\text{loc}}^1}^2), \quad b_s + b^2 \sim \frac{dJ_2}{ds} + O(\|\varepsilon\|_{H_{\text{loc}}^1}^2) \quad (1.28)$$

with

$$|J_i| \lesssim \|\varepsilon\|_{H_{\text{loc}}^1} + \int_{y>0} |\varepsilon|, \quad i = 1, 2.$$

This explains the need for a control of radiation on the right as slow tails and large J_i might otherwise perturb the formal system (1.26) (see also [18]).

(iii). *The mixed energy/Virial estimate (section 3)*

The main new input of our analysis is the derivation of a dispersive control on the local norm $\|\varepsilon\|_{H_{\text{loc}}^1}$ which is relevant in all three regimes, and therefore must display some scaling invariant structure. For this, we adapt and revisit the construction of mixed energy/Virial functionals as introduced in [22], [40], [35], [37]. Indeed, we build a nonlinear functional

$$\mathcal{F} \sim \int \psi \varepsilon_y^2 + \varphi \varepsilon^2 - \frac{1}{3} \psi [(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon]$$

for well chosen cut off functions (ψ, φ) which are exponentially decaying to the left, and polynomially growing to the right. The leading order quadratic term relates to the linearized Hamiltonian and is coercive from our choice of orthogonality conditions:

$$\mathcal{F} \gtrsim \|\varepsilon\|_{H_{\text{loc}}^1}^2.$$

The essential feature now is the structure of the cut off which is manufactured to also reproduce on the ground state the leading order virial quadratic form which measures some repulsivity properties of the linearized operator L' as derived in [17], and leads to the *Lyapounov monotonicity*:

$$\frac{d}{ds} \left\{ \frac{\mathcal{F}}{\lambda^{2j}} \right\} + \frac{\|\varepsilon\|_{H_{\text{loc}}^1}^2}{\lambda^{2j}} \lesssim \frac{|b|^4}{\lambda^{2j}}, \quad j = 0, 1. \quad (1.29)$$

The b^4 term relates to the error in the construction of the Q_b profile as an approximate solution to (1.25). The case $j = 0$ in (1.29) is a scaling invariant estimate which will be crucial in all three regimes to control the dynamics, and the case $j = 1$ is an H^1 improvement in the blow up regime $\lambda \rightarrow 0$.

(iv). *Rigidity (section 4)*

The combinaison of the modulation equations (1.28) with the dispersive bound (1.29) leads roughly speaking to⁸:

$$\frac{b(t)}{\lambda^2(t)} \sim \ell, \quad (1.30)$$

for some constant ℓ . Then the selection of the dynamics depends on:

- either $\forall t, |b(t)| \lesssim \|\varepsilon(t)\|_{H_{\text{loc}}^1}^2$,
- or there exists a time t_1^* such that $|b(t_1^*)| \gg \|\varepsilon(t_1^*)\|_{H_{\text{loc}}^1}^2$.

⁷see (2.20)

⁸see (4.14)

The second condition means that the finite dimensional dynamics measured by b takes control over the infinite dimensional dynamics at some time t_1^* . We claim that *this regime is trapped* and that $|b(t)| \gg \|\varepsilon(t)\|_{H_{10}^1}^2$ for $t \geq t_1^*$ as long as the solution remains in the tube \mathcal{T}_{α^*} . Reintegrating the modulation equations driven to leading order by (1.26), we show that this leads to (Blow up) if $b(t_1^*) > 0$ and to (Exit) if $b(t_1^*) < 0$. The first case leads to the threshold (Soliton) dynamics. The condition on $b(t_1)$ which determines the (Blow up) and (Exit) regimes is by continuity of the flow an open condition on the data.

(v). *End of the proof of Theorem 1.1*

The case $E_0 \leq 0$ is treated in section 5. here the variational characterization of Q and a standard concentration compactness ensures that the solution must remain in \mathcal{T}_{α^*} , and then we show (Blow up) by proving that (Soliton) cannot happen. For $E_0 < 0$, this is a classical consequence of the energy conservation law and local dispersive estimates (asymptotic stability) obtained in the previous step. The case $E_0 = 0$ is substantially more subtle, and we show that (Soliton) behavior at zero energy implies L^2 compactness, and hence asymptotic stability implies that the solution has minimal mass, and hence is exactly a solitary wave.

Finally, we complete in section 6 the sharp description of the singularity formation and the universality of the focusing bubble stated by Theorem 1.1. This requires propagating the dispersive estimates, which involve local norms around the soliton, further away on the left of the soliton, in particular to compute the trace of the reminder (1.18). This is done using suitable H^1 monotonicity formula in the spirit of the analysis in [24], [18].

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2. Nonlinear profiles and decomposition close to the soliton

In this section, we introduce refined nonlinear profiles following the strategy developed in [25], [35]. The strategy is to produce approximate solutions to the renormalized flow (1.25) which are as well localized as possible, which turns out to lead to a strong rigidity for the scaling law.

2.1. Structure of the linearized operator. Denote by \mathcal{Y} the set of functions $f \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\forall k \in \mathbb{N}, \exists C_k, r_k > 0, \forall y \in \mathbb{R}, |f^{(k)}(y)| \leq C_k(1 + |y|)^{r_k} e^{-|y|}. \quad (2.1)$$

We recall without proof the following standard result (see e.g. [42], [16]).

Lemma 2.1 (Properties of the linearized operator L). *The self-adjoint operator L on L^2 satisfies:*

- (i) *Eigenfunctions* : $LQ^3 = -8Q^3$; $LQ' = 0$; $\text{Ker} L = \{aQ', a \in \mathbb{R}\}$;
- (ii) *Scaling* : $L(\Lambda Q) = -2Q$;
- (iii) *For any function $h \in L^2(\mathbb{R})$ orthogonal to Q' for the L^2 scalar product, there exists a unique function $f \in H^2(\mathbb{R})$ orthogonal to Q' such that $Lf = h$; moreover, if h is even (respectively, odd), then f is even (respectively, odd).*
- (iv) *If $f \in L^2(\mathbb{R})$ is such that $Lf \in \mathcal{Y}$, then $f \in \mathcal{Y}$.*

(v) *Coercivity of L : for all $f \in H^1$,*

$$(f, Q^3) = (f, Q') = 0 \Rightarrow (Lf, f) \geq \|f\|_{L^2}^2. \quad (2.2)$$

Moreover, there exists $\mu_0 > 0$ such that for all $f \in H^1$,

$$(Lf, f) \geq \mu_0 \|f\|_{H^1}^2 - \frac{1}{\mu_0} [(\varepsilon, Q)^2 + (\varepsilon, y\Lambda Q)^2 + (\varepsilon, \Lambda Q)^2]. \quad (2.3)$$

2.2. Definition and estimates of localized profiles. We now look for a slowly modulated approximate solution to the renormalized flow (1.24), (1.25). In fact, in our setting, an order b expansion is enough.

Proposition 2.2 (Nonlocalised profiles). *There exists a unique smooth function P such that $P' \in \mathcal{Y}$ and*

$$(LP)' = \Lambda Q, \quad \lim_{y \rightarrow -\infty} P(y) = \frac{1}{2} \int Q, \quad \lim_{y \rightarrow +\infty} P(y) = 0, \quad (2.4)$$

$$(P, Q) = \frac{1}{16} \left(\int Q \right)^2 > 0, \quad (P, Q') = 0. \quad (2.5)$$

Moreover,

$$\tilde{Q}_b = Q + bP$$

is an approximate solution to (1.25) in the sense that:

$$\left\| \left(\tilde{Q}_b'' - \tilde{Q}_b + \tilde{Q}_b^5 \right)' + b\Lambda\tilde{Q}_b \right\|_{L^\infty} \lesssim b^2. \quad (2.6)$$

Proof of Proposition 2.2. We look for P of the form $P = \tilde{P} - \int_y^{+\infty} \Lambda Q$. Since $\int \Lambda Q = -\frac{1}{2} \int Q$, the function $y \mapsto \int_y^{+\infty} \Lambda Q$ is bounded and has decay only as $y \rightarrow +\infty$. Then, P solves (2.4) if

$$(L\tilde{P})' = \Lambda Q + \left(L \int_y^{+\infty} \Lambda Q \right)' = R' \quad \text{where} \quad R = (\Lambda Q)' - 5Q^4 \int_y^{+\infty} \Lambda Q.$$

Note that $R \in \mathcal{Y}$. Since $\int (\Lambda Q)Q = 0$ and $LQ' = 0$, we have $\int RQ' = -\int R'Q = 0$ and so from Lemma 2.1, there exists a unique $\tilde{P} \in \mathcal{Y}$ (and smooth), orthogonal to Q' , such that $L\tilde{P} = R$. Then $P = \tilde{P} - \int_y^{+\infty} \Lambda Q$ satisfies (2.4) and $\int PQ' = 0$. We now compute from $L(\Lambda Q) = -2Q$:

$$2 \int PQ = - \int (LP)\Lambda Q = \int \Lambda Q \int_y^{+\infty} \Lambda Q = \frac{1}{2} \left(\int \Lambda Q \right)^2 = \frac{1}{8} \left(\int Q \right)^2. \quad (2.7)$$

Finally, for $\tilde{Q}_b = Q + bP$, we have

$$\begin{aligned} & (\tilde{Q}_b'' - \tilde{Q}_b + \tilde{Q}_b^5)' + b\Lambda Q \\ &= b(-(LP)' + \Lambda Q) + b^2((10Q^3P^2)' + \Lambda P) + b^3(10Q^2P^3)' + b^4(5QP^4)' + b^5(P^5)' \end{aligned}$$

which yields (2.6). \square

Remark 2.3. Since $\int \Lambda Q = -\frac{1}{2} \int Q \neq 0$, a solution P of $(LP)' = \Lambda Q$ cannot belong to $L^2(\mathbb{R})$. We have chosen the only solution P which converges to 0 at $+\infty$ and orthogonal to Q' . The fact that P displays a non trivial tail on the left from (2.4) is an essential feature of the critical (gKdV) problem and will be central in the derivation of the blow up speed, see the proof of (2.37). Such nonlocal profile are substitute to dispersive tail (see a similar use in [20]).

We now proceed to a simple localization of the profile to avoid some artificial growth at $-\infty$. Let $\chi \in \mathcal{C}^\infty(\mathbb{R})$ be such that $0 \leq \chi \leq 1$, $\chi' \geq 0$ on \mathbb{R} , $\chi \equiv 1$ on $[-1, +\infty)$, $\chi \equiv 0$ on $(-\infty, -2]$. We fix:

$$\gamma = \frac{3}{4}, \quad (2.8)$$

(note that any $\gamma \in (2/3, 1)$ works and $3/4$ has no specific meaning here) and define the localized profile:

$$\chi_b(y) = \chi(|b|^\gamma y), \quad Q_b(y) = Q(y) + b\chi_b(y)P(y). \quad (2.9)$$

Lemma 2.4 (Definition of localized profiles and properties). *There holds for $|b| < b^*$ small enough:*

(i) Estimates on Q_b : For all $y \in \mathbb{R}$,

$$|Q_b(y)| \lesssim e^{-|y|} + |b| \left(\mathbf{1}_{[-2,0]}(|b|^\gamma y) + e^{-\frac{|y|}{2}} \right), \quad (2.10)$$

$$|Q_b^{(k)}(y)| \lesssim e^{-|y|} + |b|e^{-\frac{|y|}{2}} + |b|^{1+k\gamma} \mathbf{1}_{[-2,-1]}(|b|^\gamma y), \quad \text{for } k \geq 1. \quad (2.11)$$

where $\mathbf{1}_I$ denotes the characteristic function of the interval I .

(ii) Equation of Q_b : Let

$$-\Psi_b = (Q_b'' - Q_b + Q_b^5)' + b\Lambda Q_b. \quad (2.12)$$

Then, for all $y \in \mathbb{R}$,

$$|\Psi_b(y)| \lesssim |b|^{1+\gamma} \mathbf{1}_{[-2,-1]}(|b|^\gamma y) + b^2 \left(e^{-\frac{|y|}{2}} + \mathbf{1}_{[-2,0]}(|b|^\gamma y) \right), \quad (2.13)$$

$$|\Psi_b^{(k)}(y)| \lesssim |b|^{1+(k+1)\gamma} \mathbf{1}_{[-2,-1]}(|b|^\gamma y) + b^2 e^{-\frac{|y|}{2}}, \quad \text{for } k \geq 1. \quad (2.14)$$

(iii) Mass and energy properties of Q_b :

$$\left| \int Q_b^2 - \left(\int Q^2 + 2b \int PQ \right) \right| \lesssim |b|^{2-\gamma}, \quad (2.15)$$

$$\left| E(Q_b) + b \int PQ \right| \lesssim b^2. \quad (2.16)$$

Proof of Lemma 2.4. Proof of (i): First, from (1.2), for all $k \geq 0$, $|Q^{(k)}(y)| \lesssim e^{-|y|}$ on \mathbb{R} . Since $P' \in \mathcal{Y}$ and $\lim_{+\infty} P = 0$, we have $|P(y)| \lesssim e^{-\frac{|y|}{2}}$ for $y > 0$. Estimates (2.10) and (2.11) then follow from the definition of χ .

Proof of (ii): Expanding $Q_b = Q + b\chi_b P$ in the expression of Ψ_b and using $Q'' - Q + Q^5 = 0$, $(LP)' = \Lambda Q$, we find

$$\begin{aligned} -\Psi_b &= b(1 - \chi_b)\Lambda Q \\ &+ b((\chi_b)_{yyy}P + 3(\chi_b)_{yy}P' + 3(\chi_b)_yP'' - (\chi_b)_yP + 5(\chi_b)_yQ^4P) \\ &+ b^2((10Q^3\chi_b^2P^2)_y + P\Lambda\chi_b + \chi_b yP') \\ &+ b^3(10Q^2\chi_b^3P^3)_y + b^4(5Q\chi_b^4P^4)_y + b^5(\chi_b^5P)_y. \end{aligned} \quad (2.17)$$

Therefore, estimates (2.13) and (2.14) follow from the properties of Q , χ and P . In particular, note that:

$$|b(1 - \chi_b)\Lambda Q| \lesssim |b|e^{-\frac{3}{4}|y|} \mathbf{1}_{(-\infty, -1]}(|b|^\gamma y) \lesssim |b|e^{-\frac{|b|^{-\gamma}}{4}} e^{-\frac{|y|}{2}} \lesssim |b|^2 e^{-\frac{|y|}{2}},$$

$$b^2|P\Lambda\chi_b| \lesssim b^2(e^{-\frac{|y|}{2}} + \mathbf{1}_{[-2,-1]}(|b|^\gamma y)).$$

Proof of (iii): We first estimate from the explicit form of P :

$$\int \chi_b^2 P^2 \sim_{b \rightarrow 0} C_0^2 |b|^{-\gamma}$$

for some universal constant $C_0 > 0$. Estimate (2.15) now follows from

$$\int Q_b^2 = \int Q^2 + 2b \int \chi_b P Q + b^2 \int \chi_b^2 P^2$$

and then:

$$\int Q_b^2 \geq \int Q^2 + 2b \int P Q - C_0^2 |b|^{2-\gamma}, \quad \|Q_b - Q\|_{L^2} \sim_{b \rightarrow 0} C_0 |b|^{1-\frac{\gamma}{2}}.$$

Finally, expanding $Q_b = Q + b\chi_b P$ in $E(Q_b)$, we get

$$E(Q_b) = E(Q) - b \int \chi_b P(Q'' + Q^5) + O(b^2)$$

and using $E(Q) = 0$ and $Q'' + Q^5 = Q$ yields (2.16). \square

2.3. Decomposition of the solution using refined profiles. In this paper, we work with an H^1 solution u to (1.1) a priori in the modulated tube \mathcal{T}_{α^*} of functions near the soliton manifold. More explicitly, we assume that there exist $(\lambda_1(t), x_1(t)) \in \mathbb{R}_+^* \times \mathbb{R}$ and $\varepsilon_1(t)$ such that

$$\forall t \in [0, t_0], \quad u(t, x) = \frac{1}{\lambda_1^{\frac{1}{2}}(t)} (Q + \varepsilon_1) \left(t, \frac{x - x_1(t)}{\lambda_1(t)} \right)$$

with, $\forall t \in [0, t_0]$,

$$\|\varepsilon_1(t)\|_{L^2} \leq \kappa \leq \kappa_0 \quad (2.18)$$

for a small enough universal constant $\kappa_0 > 0$. We then have the following standard refined modulation lemma:

Lemma 2.5 (Refined modulated flow). *Assuming (2.18), there exist continuous functions $(\lambda, x, b) : [0, t_0] \rightarrow (0, +\infty) \times \mathbb{R}^2$ such that*

$$\forall t \in [0, t_0], \quad \varepsilon(t, y) = \lambda^{\frac{1}{2}}(t) u(t, \lambda(t)y + x(t)) - Q_{b(t)}(y) \quad (2.19)$$

satisfies the orthogonality conditions:

$$(\varepsilon(t), y\Lambda Q) = (\varepsilon(t), \Lambda Q) = (\varepsilon(t), Q) = 0. \quad (2.20)$$

Moreover,

$$\|\varepsilon(t)\|_{L^2} + |b(t)| + \left| 1 - \frac{\lambda(t)}{\lambda_1(t)} \right| \lesssim \delta(\kappa), \quad \|\varepsilon(t)\|_{H^1} \lesssim \delta(\|\varepsilon(0)\|_{H^1}). \quad (2.21)$$

Remark 2.6. The main novelty here with respect to [17], [24], [18] is the use of the modulation parameter b which allows for the extra degeneracy $(\varepsilon, Q) = 0$. At the formal level, the parameter b now plays the role of (ε, Q) for the previous work [18].

Proof. Lemma 2.5 is a standard consequence of the implicit function theorem applied in L^2 . We omit the details and refer for example to [25] for a proof with similar Q_b profiles for the (NLS) case. The heart of the proof is the non degeneracy of the Jacobian matrix:

$$\begin{vmatrix} (\Lambda Q, \Lambda Q) & (\Lambda Q, Q) \\ (P, \Lambda Q) & (P, Q) \end{vmatrix} = (\Lambda Q, \Lambda Q)(P, Q) \neq 0,$$

from

$$\frac{\partial}{\partial \lambda} \left\{ \lambda^{1/2} Q_b(\lambda y) \right\}_{|\lambda=1, b=0} = \Lambda Q, \quad \frac{\partial}{\partial b} \left\{ \lambda^{1/2} Q_b(\lambda y) \right\}_{|\lambda=1, b=0} = P,$$

and the explicit computations

$$(\Lambda Q, Q) = 0, \quad (P, Q) = \frac{1}{16} \int Q^2 \neq 0.$$

□

2.4. Modulation equations. In the framework of Lemma 2.5, we introduce the new time variable

$$s = \int_0^t \frac{dt'}{\lambda^3(t')} \quad \text{or equivalently} \quad \frac{ds}{dt} = \frac{1}{\lambda^3}. \quad (2.22)$$

All functions depending on $t \in [0, t_0]$, for some $t_0 > 0$ can now be seen as depending on $s \in [0, s_0]$, where $s_0 = s(t_0)$. We now claim the following properties of the decomposition of $u(t)$, possibly taking a smaller universal $\kappa_0 > 0$.

Lemma 2.7 (Modulation equations). *Assume for all $t \in [0, t_0]$,*

$$\|\varepsilon(t)\|_{L^2} \leq \kappa \leq \kappa_0 \quad \text{and} \quad \int \varepsilon_y^2(t, y) e^{-\frac{|y|}{2}} dy \leq \kappa_0 \quad (2.23)$$

for a small enough universal constant $\kappa_0 > 0$. Then the map $s \in [0, s_0] \mapsto (\lambda(s), x(s), b(s))$ is \mathcal{C}^1 and the following holds:

(i) Equation of ε : For all $s \in [0, s_0]$,

$$\begin{aligned} \varepsilon_s - (L\varepsilon)_y + b\Lambda\varepsilon &= \left(\frac{\lambda_s}{\lambda} + b \right) (\Lambda Q_b + \Lambda\varepsilon) + \left(\frac{x_s}{\lambda} - 1 \right) (Q_b + \varepsilon)_y \\ &\quad + \Phi_b + \Psi_b - (R_b(\varepsilon))_y - (R_{\text{NL}}(\varepsilon))_y, \end{aligned} \quad (2.24)$$

where Ψ_b is defined in (2.12) and

$$\Phi_b = -b_s (\chi_b + \gamma y(\chi_b)_y) P, \quad (2.25)$$

$$R_b(\varepsilon) = 5(Q_b^4 - Q^4)\varepsilon, \quad R_{\text{NL}}(\varepsilon) = (\varepsilon + Q_b)^5 - 5Q_b^4\varepsilon - Q_b^5. \quad (2.26)$$

(ii) Estimates induced by the conservation laws: on $[0, s_0]$, there holds

$$\|\varepsilon\|_{L^2}^2 \lesssim |b|^{\frac{1}{2}} + \left| \int u_0^2 - \int Q^2 \right|, \quad (2.27)$$

$$\left| 2\lambda^2 E_0 + \frac{b}{8} \|Q\|_{L^1}^2 - \|\varepsilon_y\|_{L^2}^2 \right| \lesssim b^2 + \|\varepsilon(s)\|_{L^2}^2 + \delta(\|\varepsilon\|_{L^2}) \|\varepsilon_y\|_{L^2}^2. \quad (2.28)$$

(iii) H^1 modulation equations: for all $s \in [0, s_0]$,

$$\left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \lesssim \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + b^2; \quad (2.29)$$

$$|b_s| \lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + b^2. \quad (2.30)$$

(iv) Refined modulation equations in \mathcal{A} : Assuming the following uniform L^1 control on the right:

$$\forall t \in [0, t_0), \quad \int_{y>0} |\varepsilon(t)| \lesssim \delta(\kappa_0), \quad (2.31)$$

then the quantities J_1 and J_2 below are well-defined and satisfy the following:

- *Law of λ : let*

$$\rho_1(y) = \frac{4}{\left(\int Q\right)^2} \int_{-\infty}^y \Lambda Q, \quad J_1(s) = (\varepsilon(s), \rho_1), \quad (2.32)$$

then for some universal constant c_1 ,

$$\left| \frac{\lambda_s}{\lambda} + b + c_1 b^2 - 2 \left((J_1)_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J_1 \right) \right| \lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + |b| \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b|^3. \quad (2.33)$$

- *Law of b : let*

$$\rho_2 = \frac{16}{\left(\int Q\right)^2} \left(\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} \Lambda Q + P - \frac{1}{2} \int Q \right) - 8\rho_1, \quad J_2(s) = (\varepsilon(s), \rho_2), \quad (2.34)$$

then for some universal constant c_2 .

$$\left| b_s + 2b^2 + c_2 b^3 + b \left((J_2)_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J_2 \right) \right| \lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + |b|^4. \quad (2.35)$$

- *Law of $\frac{b}{\lambda^2}$: let*

$$\rho = 4\rho_1 + \rho_2 \in \mathcal{Y}, \quad J(s) = (\varepsilon(s), \rho), \quad (2.36)$$

then, for $c_0 = c_2 - 2c_1$,

$$\left| \frac{d}{ds} \left(\frac{b}{\lambda^2} \right) + \frac{b}{\lambda^2} \left(J_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J \right) + c_0 \frac{b^3}{\lambda^2} \right| \lesssim \frac{1}{\lambda^2} \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} + |b|^4 \right). \quad (2.37)$$

Remark 2.8. It is a remarkable algebraic fact that the equation of $\frac{b}{\lambda^2}$ (2.37) is related to $\rho \in \mathcal{Y}$ which means that J is an L^2 quantity, easier to control than J_1 and J_2 separately.

The equations (2.33), (2.35) correspond to a sharp improvement – after integration in time – of the rough estimates of (iii). However, they hold for initial data in weighted spaces such as \mathcal{A} . Here we are facing an intrinsic difficulty of the (gKdV) equation which is that the null space of the full linearized operators $(L)'$ involves badly localized terms, and hence getting geometrical parameters which are quadratic forcing terms of the ε equation (2.24) requires some L^1 control of the solution on the right. Formally, (2.33), (2.35) are the sharp analogues of the leading order dynamical system:

$$\frac{\lambda_s}{\lambda} = -b, \quad \left(\frac{b}{\lambda^2} \right)_s = \frac{b_s + 2b^2}{\lambda^3} = 0.$$

Proof of Lemma 2.7. Proof of (i): The equation of $\varepsilon, \lambda, x, b$ follows by direct computations from the equation of $u(t)$. In particular, we use

$$\frac{\partial}{\partial b}(Q_b) = \frac{\partial}{\partial b}(b\chi_b(y))P = (\gamma|b|^\gamma y\chi'(|b|^\gamma y) + \chi(|b|^\gamma y))P = (\chi_b + \gamma y(\chi_b)_y)P.$$

The rest of the computation is done in Lemma 1 of [16] for example.

Proof of (ii): We write down the L^2 conservation law:

$$\int Q_b^2 - \int Q^2 + \int \varepsilon^2 + 2(\varepsilon, Q_b) = \int u_0^2 - \int Q^2$$

and we deduce from (2.15) using the orthogonality condition (2.20) that

$$\int \varepsilon^2 \lesssim |b| + |b|^{1-\gamma} \|\varepsilon\|_{L^2} + \left| \int u_0^2 - \int Q^2 \right|.$$

Then (2.27) follows since $\gamma = 3/4$.

Now, we write down the conservation of energy and use (2.16), the equation of Q and the orthogonality condition $(\varepsilon, Q) = 0$ to estimate:

$$\begin{aligned} 2\lambda^2 E(u_0) &= 2E(Q_b) - 2 \int \varepsilon(Q_b)_{yy} + \int \varepsilon_y^2 - \frac{1}{3} \int ((Q_b + \varepsilon)^6 - Q_b^6) \\ &= -2b(P, Q) + O(b^2) + \int \varepsilon_y^2 \\ &\quad - 2 \int \varepsilon [(Q_b - Q)_{yy} + (Q_b^5 - Q^5)] \\ &\quad - \frac{1}{3} \int [(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon]. \end{aligned}$$

We estimate all terms in the above identity. By the properties of Q_b :

$$\begin{aligned} \left| \int \varepsilon [(Q_b - Q)_{yy} + (Q_b^5 - Q^5)] \right| &\lesssim |b| \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b|^{1+2\gamma} \int_{-2|b|^{-\gamma} < y < 0} |\varepsilon| \\ &\lesssim b^2 + \|\varepsilon\|_{L^2}^2. \end{aligned}$$

The nonlinear terms are estimated by the homogeneity of the nonlinearity which implies:

$$\begin{aligned} \left| \int [(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon] \right| &\lesssim \int |Q_b|^4 \varepsilon^2 + |\varepsilon|^6 \\ &\lesssim \|\varepsilon\|_{L^2}^2 + \|\varepsilon_y\|_{L^2}^2 \|\varepsilon\|_{L^2}^4. \end{aligned}$$

The collection of above estimates yields (2.28).

Proof of (iii): We sketch the standard computations⁹ leading to (2.29) and (2.30). Differentiating the orthogonality conditions $(\varepsilon, \Lambda Q) = (\varepsilon, y\Lambda Q) = 0$, using the equation of ε and estimate (2.13), we obtain:

$$\begin{aligned} &\left| \left(\frac{\lambda_s}{\lambda} + b \right) - \frac{(\varepsilon, L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2} \right| + \left| \left(\frac{x_s}{\lambda} - 1 \right) - \frac{(\varepsilon, L(y\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2} \right| \\ &\lesssim \left(\left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| + |b| \right) \left(|b| + \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} \right) \\ &\quad + |b_s| + \int \varepsilon^2 e^{-\frac{|y|}{10}} + \int |\varepsilon|^5 e^{-\frac{9}{10}|y|}. \end{aligned}$$

We estimate the nonlinear term using the Sobolev bound¹⁰ and the smallness (2.23):

$$\|\varepsilon e^{-\frac{|y|}{4}}\|_{L^\infty}^2 \lesssim \int (|\partial_y \varepsilon|^2 + |\varepsilon|^2) e^{-\frac{|y|}{2}},$$

so that

$$\int |\varepsilon|^5 e^{-\frac{9}{10}|y|} \lesssim \|\varepsilon e^{-\frac{|y|}{4}}\|_{L^\infty}^3 \int \varepsilon^2 e^{-\frac{|y|}{10}} \quad (2.38)$$

Thus, (2.23), and for κ_0 small enough,

$$\begin{aligned} &\left| \left(\frac{\lambda_s}{\lambda} + b \right) - \frac{(\varepsilon, L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2} \right| + \left| \left(\frac{x_s}{\lambda} - 1 \right) - \frac{(\varepsilon, L(y\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2} \right| \\ &\lesssim |b|^2 + |b_s| + \int \varepsilon^2 e^{-\frac{|y|}{10}}, \end{aligned} \quad (2.39)$$

⁹See e.g. [16], Lemma 4, for similar computations.

¹⁰which follows by integration by parts.

and

$$\left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \lesssim |b|^2 + |b_s| + \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}}. \quad (2.40)$$

Next, differentiating in time s the relation $(\varepsilon, Q) = 0$, using the ε equation, and the following algebraic facts $LQ' = 0$, $(Q, \Lambda Q) = (Q, Q') = 0$, $(\varepsilon, \Lambda Q) = 0$, the nondegeneracy $(P, Q) \neq 0$ and the bounds (2.13), (2.14), we find after integration by parts and Sobolev estimates (2.38):

$$|b_s| \lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2 + \left| \frac{x_s}{\lambda} - 1 \right|^2 + |b|^2 + \int \varepsilon^2 e^{-\frac{|y|}{10}}. \quad (2.41)$$

(see below a much detailed computation of b_s).

Combining (2.40) and (2.41) yields (2.29), (2.30).

Proof of (iv): To begin, we claim the following sharp equation for b :

$$\begin{aligned} b_s + 2b^2 + cb^3 - \frac{16}{\left(\int Q\right)^2} b \left[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} (\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3Q') \right] \\ = O(|b|^4) + O\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right), \end{aligned} \quad (2.42)$$

where c is a universal constant.

To prove (2.42), we take the scalar product of the equation of ε by Q and we keep track of all terms up to order $|b|^3$.

In this proof, c will denote various universal constants. First, we use the explicit formula (2.17) to derive:

$$\begin{aligned} (\Psi_b, Q) &= -b^2 ((10Q^3 \chi_b^2 P^2)_y + \chi_b \Lambda P, Q) - b^3 (10Q^2 \chi_b^3 P^3, Q') + O(|b|^4) \\ &= -b^2 ((10P^2 Q^3)' + \Lambda P, Q) - b^3 (10Q^2 P^3, Q') + O(|b|^4) \\ &= -\frac{b^2}{8} \|Q\|_{L^1}^2 + c_0 b^3 + O(|b|^4), \end{aligned}$$

where $c_0 = -10 \int P^3 Q^2 Q'$ and where we have used in the last step the following fundamental *flux computation*:

$$\begin{aligned} (\Lambda P, Q) &= -(P, \Lambda Q) = -(P, (LP)') = (P, (P'' - P + 5Q^4 P)') \\ &= (P, P''' - P') + 10 \int Q^3 Q' P^2 \end{aligned}$$

from which we indeed obtain

$$((10P^2 Q^3)' + \Lambda P, Q) = \frac{1}{2} \lim_{-\infty} P^2 = \frac{1}{8} \left(\int Q \right)^2. \quad (2.43)$$

This computation is the key to the derivation of the blow up speed.

From (2.5):

$$(\Phi_b, Q) = -(b_s(\chi_b + \gamma y \chi_b') P, Q) = -b_s(P, Q) + O(b^{10}) = -\frac{b_s}{16} \left(\int Q \right)^2 + O(b^{10}).$$

Next from (2.5):

$$\left| \left(\frac{x_s}{\lambda} - 1 \right) (Q_b, Q') \right| + \left| \int (\Lambda Q_b) Q - b(\Lambda P, Q) \right| \lesssim |b|^{10}.$$

We estimate the small linear term as follows

$$\int R_b(\varepsilon) Q' = 20b \int PQ^3 Q' \varepsilon + b^2 O\left(\int \varepsilon^2 e^{-\frac{|y|}{10}}\right)^{\frac{1}{2}},$$

and nonlinear terms in ε are simply treated as before by (2.38).

Therefore, we have obtained:

$$\begin{aligned} b_s + 2b^2 + cb^3 - \frac{16}{(\int Q)^2} b \left[\left(\frac{\lambda_s}{\lambda} + b \right) (\Lambda P, Q) + 20(\varepsilon, PQ^3Q') \right] \\ = O(|b|^4) + O \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right), \end{aligned} \quad (2.44)$$

Moreover, we check that when estimating $\frac{\lambda_s}{\lambda} + b$, using

$$|b_s + 2b^2| \leq |b|^3 + |b| \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + \int \varepsilon^2 e^{-\frac{|y|}{10}},$$

and keep track of all b^2 terms, we can improve (2.39) into

$$\left| \left(\frac{\lambda_s}{\lambda} + b \right) - \frac{(\varepsilon, L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2} - cb^2 \right| \lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + |b| \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} + |b|^3. \quad (2.45)$$

Estimate (2.42) follows from (2.44) and (2.45).

Thanks to the L^1 bound (2.31), for any $f \in \mathcal{Y}$, $(\varepsilon, \int_{-\infty}^y f)$ is well defined for all time and by direct computations, we have the following general formula:

$$\begin{aligned} \frac{d}{ds} \left(\varepsilon, \int_{-\infty}^y f \right) &= -(\varepsilon, Lf) + \left(\frac{\lambda_s}{\lambda} + b \right) \left(\Lambda Q_b, \int_{-\infty}^y f \right) + \frac{\lambda_s}{\lambda} \left(\Lambda \varepsilon, \int_{-\infty}^y f \right) \\ &\quad - \left(\frac{x_s}{\lambda} - 1 \right) (Q_b, f) - \left(\frac{x_s}{\lambda} - 1 \right) (\varepsilon, f) - b_s \left((\chi_b + \gamma y \chi'_b) P, \int_{-\infty}^y f \right) \\ &\quad + \left(\Psi_b, \int_{-\infty}^y f \right) + (R_b(\varepsilon) + R_{NL}(\varepsilon), f). \end{aligned} \quad (2.46)$$

Using (2.29), (2.30), (2.13) and (2.42), we obtain from (2.46):

$$\begin{aligned} \frac{d}{ds} \left(\varepsilon, \int_{-\infty}^y f \right) &= -(\varepsilon, Lf) + \left(\frac{\lambda_s}{\lambda} + b \right) \left(\Lambda Q, \int_{-\infty}^y f \right) - \left(\frac{x_s}{\lambda} - 1 \right) (f, Q) \\ &\quad - \frac{1}{2} \frac{\lambda_s}{\lambda} \left(\varepsilon, \int_{-\infty}^y f \right) + cb^2 + O \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right) \\ &\quad + O \left(|b| \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} \right) + O(|b|^3), \end{aligned} \quad (2.47)$$

for some constant c depending on f .

– Equation of J_1 : We apply (2.47) to $f = \Lambda Q$ using the following algebraic relations

$$L\Lambda Q = -2Q, \quad \left(\Lambda Q, \int_{-\infty}^y \Lambda Q \right) = \frac{1}{8} \left(\int Q \right)^2, \quad \left(Q', \int_{-\infty}^y \Lambda Q \right) = 0,$$

to prove

$$\begin{aligned} 2(J_1)_s &= \frac{16(\varepsilon, Q)}{(\int Q)^2} + \left(\frac{\lambda_s}{\lambda} + b \right) - \frac{\lambda_s}{\lambda} J_1 + cb^2 \\ &\quad + O \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right) + O \left(|b| \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} \right) + O(|b|^3). \end{aligned}$$

The orthogonality conditions (2.20) now yield (2.33).

– Equation of J_2 . We now apply (2.47) to $\int_{-\infty}^y f = \rho_2$, $f = \rho'_2$. We need some computation related to ρ_2 . Using $\int \Lambda Q = -\frac{1}{2} \int Q$,

$$\begin{aligned} (\Lambda Q, \rho_2) &= \frac{16}{(\int Q)^2} \left(\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} \Lambda Q + P - \frac{1}{2} \int Q, \Lambda Q \right) - \frac{32}{(\int Q)^2} (\Lambda Q, \int_{-\infty}^y \Lambda Q) \\ &= \frac{16}{(\int Q)^2} [(\Lambda P, Q) + (\Lambda Q, P)] + \frac{4}{(\int Q)^2} (\int Q)^2 - \frac{16}{(\int Q)^2} (\int \Lambda Q)^2 = 0, \end{aligned}$$

and similarly:

$$(\rho'_2, Q) = \frac{16}{(\int Q)^2} \left(\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} (\Lambda Q)' + P', Q \right) - 8(\rho'_1, Q) = 0.$$

Next the algebra

$$L(P') = (LP')' + 20Q^3 Q' P = \Lambda Q + 20Q^3 Q' P,$$

and the orthogonality relations $(\varepsilon, \Lambda Q) = 0$, $(P, Q') = 0$ yield:

$$\begin{aligned} (\varepsilon, (L\rho'_2)) &= \frac{16}{(\int Q)^2} \left(\varepsilon, L \left[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} (\Lambda Q)' + P' \right] \right) - 8(\varepsilon, L\rho'_1) \\ &= \frac{16}{(\int Q)^2} \left[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} (\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3 Q') \right]. \end{aligned}$$

Injecting these relations into (2.46) yields:

$$\begin{aligned} \frac{d}{ds} J_2 &= -\frac{16}{(\int Q)^2} \left[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2} (\varepsilon, L(\Lambda Q)') + 20 \int \varepsilon PQ^3 Q' \right] - \frac{1}{2} \frac{\lambda_s}{\lambda} J_2 \\ &\quad + cb^2 + O \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right) + O \left(|b| \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} \right)^{\frac{1}{2}} \right) + O(|b|^3). \end{aligned} \quad (2.48)$$

Combining (2.42) and (2.48) yields (2.35).

– Equation of J . We now compute from (2.33) and (2.35):

$$\begin{aligned} \frac{d}{ds} \left(\frac{b}{\lambda^2} \right) &= \frac{b_s}{\lambda^2} - 2 \frac{\lambda_s}{\lambda} \frac{b}{\lambda^2} = \frac{b_s + 2b^2}{\lambda^2} - \frac{2b}{\lambda^2} \left(\frac{\lambda_s}{\lambda} + b \right) \\ &= -\frac{b}{\lambda^2} \left[(J_2)_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J_2 \right] - \frac{2b}{\lambda^2} \left[2(J_1)_s + \frac{\lambda_s}{\lambda} J_1 \right] + (2c_1 - c_2) \frac{b^3}{\lambda^2} \\ &\quad + \frac{1}{\lambda^2} O \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} + b^4 \right) \\ &= -\frac{b}{\lambda^2} \left[J_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J \right] + (2c_1 - c_2) \frac{b^3}{\lambda^2} + \frac{1}{\lambda^2} O \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} + b^4 \right), \end{aligned}$$

which is (2.37).

Finally, we check that $\rho = 4\rho_1 + \rho_2 \in \mathcal{Y}$. Indeed, ρ_1, ρ_2 are exponentially localized at $-\infty$ from (2.4). We thus need only check that $\lim_{+\infty} \rho = 0$, but it is immediate from their definitions that $\lim_{+\infty} \rho_1 = -\frac{2}{fQ}$ and $\lim_{+\infty} \rho_2 = \frac{8}{fQ}$.

This concludes the proof of Lemma 2.7. \square

2.5. Kato type identities. We recall the following standard identities which correspond to the localization of conservation laws.

Claim 1 (Kato localization identities). *Let g be any C^3 function and $v(t, x)$ be a solution of (1.1). Then*

(1) L^2 identity:

$$\frac{d}{dt} \int v^2 g = -3 \int v_x^2 g' + \int v^2 g''' + \frac{5}{3} \int v^6 g'. \quad (2.49)$$

(2) Energy identity:

$$\begin{aligned} & \frac{d}{dt} \int \left(v_x^2 - \frac{1}{3} v^6 \right) g \\ &= - \int (v_{xx} + v^5)^2 g' - 2 \int v_{xx}^2 g' + 10 \int v^4 v_x^2 g' + \int v_x^2 g'''. \end{aligned} \quad (2.50)$$

3. Monotonicity formulas

This section is devoted to the derivation of the monotonicity tools for solutions near the soliton manifold which are the key technical arguments of our analysis for initial data in \mathcal{A} . We exhibit a Lyapounov functional based on a suitable localization of the linearized Hamiltonian, which will both control pointwise dispersion around the soliton, and display some monotonicity thanks to the coercivity of the virial quadratic form proved in [17]. A related strategy originated in [22], [30], [35], [31], but is implemented here in a new optimal way. Such dispersive estimates coupled with the modulation equation for b will lead to the key rigidity property for the proof of the main results of this paper.

3.1. Pointwise monotonicity. Let $(\varphi_i)_{i=1,2}, \psi \in C^\infty(\mathbb{R})$ be such that:

$$\varphi_i(y) = \begin{cases} e^y & \text{for } y < -1, \\ 1 + y & \text{for } -\frac{1}{2} < y < \frac{1}{2}, \\ y^i & \text{for } y > 2, \end{cases} \quad \varphi'_i(y) > 0, \quad \forall y \in \mathbb{R}, \quad (3.1)$$

$$\psi(y) = \begin{cases} e^{2y} & \text{for } y < -1, \\ 1 & \text{for } y > -\frac{1}{2}, \end{cases} \quad \psi'(y) \geq 0 \quad \forall y \in \mathbb{R}. \quad (3.2)$$

Let $B > 100$ be a large universal constant to be chosen in Proposition 3.1, let

$$\psi_B(y) = \psi\left(\frac{y}{B}\right), \quad \varphi_{i,B} = \varphi_i\left(\frac{y}{B}\right), \quad i = 1, 2,$$

and define the following norms on ε

$$\mathcal{N}_i(s) = \int \varepsilon_y^2(s, y) \psi_B(y) dy + \int \varepsilon^2(s, y) \varphi_{i,B}(y) dy, \quad i = 1, 2. \quad (3.3)$$

We also define the following L^2 weighted norm for ε

$$\mathcal{N}_{i,\text{loc}}(s) = \int \varepsilon^2(s, y) \varphi'_{i,B}(y) dy, \quad i = 1, 2. \quad (3.4)$$

The heart of our analysis is the following monotonicity property:

Proposition 3.1 (Monotonicity formula). *There exist $\mu > 0$, $B > 100$ and $0 < \kappa^* < \kappa_0$ such that the following holds. Assume that $u(t)$ is a solution of (1.1) which satisfies (2.18) on $[0, t_0]$ and thus admits on $[0, t_0]$ a decomposition (2.19) as in*

Lemma 2.5. Let $s_0 = s(t_0)$, and assume the following a priori bounds: $\forall s \in [0, s_0]$,
(H1) smallness:

$$\|\varepsilon(s)\|_{L^2} + |b(s)| + \mathcal{N}_2(s) \leq \kappa^*; \quad (3.5)$$

(H2) bound related to scaling:

$$\frac{|b(s)| + \mathcal{N}_2(s)}{\lambda^2(s)} \leq \kappa^*; \quad (3.6)$$

(H3) L^2 weighted bound on the right:

$$\int_{y>0} y^{10} \varepsilon^2(s, x) dx \leq 10 \left(1 + \frac{1}{\lambda^{10}(s)} \right). \quad (3.7)$$

Let the energy-virial Lyapounov functionals for $(i, j) \in \{1, 2\}^2$

$$\mathcal{F}_{i,j} = \int \left[\varepsilon_y^2 \psi_B + \varepsilon^2 (1 + \mathcal{J}_{i,j}) \varphi_{i,B} - \frac{1}{3} ((\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5) \psi_B \right], \quad (3.8)$$

with

$$\mathcal{J}_{i,j} = (1 - J_1)^{-(4(j-1)+2i)} - 1. \quad (3.9)$$

Then the following estimates hold on $[0, s_0]$:

(i) Scaling invariant Lyapounov control: for $i = 1, 2$,

$$\frac{d\mathcal{F}_{i,1}}{ds} + \mu \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} \lesssim |b|^4. \quad (3.10)$$

(ii) Scaling weighted H^1 Lyapounov control: for $i = 1, 2$,

$$\frac{d}{ds} \left\{ \frac{\mathcal{F}_{i,2}}{\lambda^2} \right\} + \frac{\mu}{\lambda^2} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} \lesssim \frac{|b|^4}{\lambda^2}. \quad (3.11)$$

(iii) Coercivity of $\mathcal{F}_{i,j}$ and pointwise bounds: there holds for $(i, j) \in \{1, 2\}^2$,

$$\mathcal{N}_i \lesssim \mathcal{F}_{i,j} \lesssim \mathcal{N}_i, \quad (3.12)$$

$$|J_i| + |\mathcal{J}_{i,j}| \lesssim \mathcal{N}_2^{\frac{1}{2}}. \quad (3.13)$$

Remark 3.2. The L^2 weighted bound (3.7) is fundamental for the analysis and will be further dynamically bootstrapped for an initial data in \mathcal{A} . Also one should think of (3.10) as a scaling invariant L^2 bound, which is sharpened in the singular regime $\lambda \rightarrow 0$ by the H^1 control (3.11). Finally, an important feature of Proposition 3.1 is that we do not assume any a priori control on the scaling parameter $\lambda(s)$.

We will use several times in the proof the fact that in the definition of $\mathcal{F}_{i,j}$, the weight on ε_y at $-\infty$ is stronger than the weight on ε . It follows in particular that $\mathcal{F}_{i,j}$ does not control $\int \varepsilon_y^2 \varphi'_{i,B}$. See Remark 3.5 below.

Proof of Proposition 3.1. step 1 Weighted L^2 controls at the right.

We first claim the controls for all $s \in [0, s_0]$,

$$\int_{y>0} y \varepsilon^2(s) \lesssim \left(1 + \frac{1}{\lambda^{\frac{10}{9}}(s)} \right) \mathcal{N}_{1,\text{loc}}^{\frac{8}{9}}(s), \quad (3.14)$$

$$\int_{y>0} y^2 \varepsilon^2(s) \lesssim \left(1 + \frac{1}{\lambda^{\frac{10}{9}}(s)} \right) \mathcal{N}_{2,\text{loc}}^{\frac{8}{9}}(s), \quad (3.15)$$

$$\int_{y>0} |\varepsilon(s)| \lesssim \mathcal{N}_2^{\frac{1}{2}}(s). \quad (3.16)$$

From (3.7): for all $A > 0$,

$$\int_{y>0} y\varepsilon^2 \leq A \int_{0 \leq y \leq A} |\varepsilon|^2 + \frac{1}{A^9} \int_{y>A} y^{10} |\varepsilon|^2 \lesssim A \mathcal{N}_{1,\text{loc}} + \frac{1}{A^9} \left(1 + \frac{1}{\lambda^{10}}\right)$$

and so the optimal choice

$$A^{10} \mathcal{N}_{1,\text{loc}} = 1 + \frac{1}{\lambda^{10}}$$

leads to the bound using the smallness (3.5):

$$\int_{y>0} y\varepsilon^2 \lesssim \frac{(1 + \lambda^{10})^{\frac{1}{10}}}{\lambda} \mathcal{N}_{1,\text{loc}}^{\frac{9}{10}} \lesssim \left(1 + \frac{1}{\lambda}\right) \mathcal{N}_{1,\text{loc}}^{\frac{9}{10}} \lesssim \left(1 + \frac{1}{\lambda^{\frac{10}{9}}}\right) \mathcal{N}_{1,\text{loc}}^{\frac{8}{9}},$$

and (3.14) is proved. Similarly,

$$\int_{y>0} y^2 \varepsilon^2 \leq A \int_{0 \leq y \leq A} y |\varepsilon|^2 + \frac{1}{A^8} \int_{y>A} y^{10} |\varepsilon|^2 \lesssim A \mathcal{N}_{2,\text{loc}} + \frac{1}{A^8} \left(1 + \frac{1}{\lambda^{10}}\right),$$

and thus the choice

$$A^9 \mathcal{N}_{2,\text{loc}} = 1 + \frac{1}{\lambda^{10}}$$

leads to the bound:

$$\int_{y>0} y^2 \varepsilon^2 \lesssim \mathcal{N}_{2,\text{loc}}^{\frac{8}{9}} \frac{(1 + \lambda^{10})^{\frac{1}{9}}}{\lambda^{\frac{10}{9}}} \lesssim \left(1 + \frac{1}{\lambda^{\frac{10}{9}}}\right) \mathcal{N}_{2,\text{loc}}^{\frac{8}{9}},$$

and (3.15) is proved.

(3.16) follows from

$$\int_{y>0} |\varepsilon| \lesssim \|(1+y)\varepsilon\|_{L^2(y>0)} \lesssim \mathcal{N}_2^{\frac{1}{2}}.$$

Finally, we observe that (3.16) implies (3.13). In particular, the quantities $\mathcal{J}_{i,j}$ are well defined, and so are $\mathcal{F}_{i,j}$.

step 2 Algebraic computations on $\mathcal{F}_{i,j}$. We compute

$$\begin{aligned} & \lambda^{2(j-1)} \frac{d}{ds} \left\{ \frac{\mathcal{F}_{i,j}}{\lambda^{2(j-1)}} \right\} \\ &= 2 \int \psi_B(\varepsilon_y)_s \varepsilon_y + 2\varepsilon_s \left[(1 + \mathcal{J}_{i,j}) \varepsilon \varphi_{i,B} - \psi_B [(\varepsilon + Q_b)^5 - Q_b^5] \right] \\ &+ (\mathcal{J}_{i,j})_s \int \varphi_{i,B} \varepsilon^2 - 2 \int \psi_B(Q_b)_s [(\varepsilon + Q_b)^5 - Q_b^5 - 5\varepsilon Q_b^4] \\ &- 2(j-1) \frac{\lambda_s}{\lambda} \mathcal{F}_{i,j} \end{aligned}$$

which we rewrite

$$\lambda^{2(j-1)} \frac{d}{ds} \left\{ \frac{\mathcal{F}_{i,j}}{\lambda^{2(j-1)}} \right\} = f_1^{(i)} + f_2^{(i,j)} + f_3^{(i,j)} + f_4^{(i)}, \quad (3.17)$$

where

$$\begin{aligned} f_1^{(i)} &= 2 \int \left(\varepsilon_s - \frac{\lambda_s}{\lambda} \Lambda \varepsilon \right) \left(-(\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B [(\varepsilon + Q_b)^5 - Q_b^5] \right), \\ f_2^{(i,j)} &= 2 \int \left(\varepsilon_s - \frac{\lambda_s}{\lambda} \Lambda \varepsilon \right) \varepsilon \mathcal{J}_{i,j} \varphi_{i,B}, \\ f_3^{(i,j)} &= 2 \frac{\lambda_s}{\lambda} \int \Lambda \varepsilon \left(-(\psi_B \varepsilon_y)_y + (1 + \mathcal{J}_{i,j}) \varepsilon \varphi_{i,B} - \psi_B [(\varepsilon + Q_b)^5 - Q_b^5] \right) \end{aligned}$$

$$\begin{aligned}
& + (\mathcal{J}_{i,j})_s \int \varphi_{i,B} \varepsilon^2 - 2(j-1) \frac{\lambda_s}{\lambda} \mathcal{F}_{i,j} \\
f_4^{(i)} & = -2 \int \psi_B (Q_b)_s ((\varepsilon + Q_b)^5 - Q_b^5 - 5\varepsilon Q_b^4).
\end{aligned}$$

We claim the following estimates on the above terms: for some $\mu_0 > 0$,

$$\frac{d}{ds} f_1^{(i)} \leq -\mu_0 \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} + C|b|^4, \quad (3.18)$$

$$\left| \frac{d}{ds} f_k^{(i)} \right| \leq \frac{\mu_0}{10} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} + C|b|^4, \quad \text{for } k = 2, 3, 4. \quad (3.19)$$

Note that in (3.18), we obtained a negative term $-\mu \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}$, related both to the smoothing effect of the (gKdV) equation and to a Virial estimate for the linearization of the (gKdV) equation close to the soliton. Inserting (3.18) and (3.19) into (3.17) indeed yields (3.10), (3.11).

In steps 3 - step 6, we prove (3.18) and (3.19). Observe that the definitions of φ_i and ψ imply the following estimates:

$$\forall y \in \mathbb{R}, \quad |\varphi_i'''(y)| + |\varphi_i''(y)| + |\psi'''(y)| + |y\psi'(y)| + |\psi(y)| \lesssim \varphi_i'(y) \lesssim \varphi_i(y), \quad (3.20)$$

$$\forall y \in (-\infty, 2], \quad e^{|y|}\psi(y) + e^{|y|}\psi'(y) + \varphi_i(y) \lesssim \varphi_i'(y), \quad (3.21)$$

$$\forall y \in \mathbb{R}, \quad \varphi_2'(y) \lesssim \varphi_1(y) \lesssim \varphi_2'(y). \quad (3.22)$$

In particular,

$$\mathcal{N}_{1,\text{loc}}(s) \lesssim \mathcal{N}_{2,\text{loc}}(s) \lesssim \mathcal{N}_1(s) \lesssim \mathcal{N}_2(s), \quad \int \varepsilon^2(s, y) \varphi_{1,B}(y) dy \lesssim \mathcal{N}_{2,\text{loc}}(s). \quad (3.23)$$

step 3 Control of $f_1^{(i)}$. Proof of (3.18). We compute $f_1^{(i)}$ using the ε equation (2.24) in the following form:

$$\begin{aligned}
\varepsilon_s - \frac{\lambda_s}{\lambda} \Lambda \varepsilon & = (-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)^5 + Q_b^5)_y \\
& + \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda Q_b + \left(\frac{x_s}{\lambda} - 1 \right) (Q_b + \varepsilon)_y + \Phi_b + \Psi_b,
\end{aligned} \quad (3.24)$$

where $\Phi_b = -b_s (\chi_b + \gamma y(\chi_b)_y) P$ and $-\Psi_b = (Q_b'' - Q_b + Q_b^5)' + b \Lambda Q_b$. This yields:

$$\begin{aligned}
f_1^{(i)} & = 2 \int (-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5))_y (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B ((Q_b + \varepsilon)^5 - Q_b^5) \\
& + 2 \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_b (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B ((\varepsilon + Q_b)^5 - Q_b^5) \\
& + 2 \left(\frac{x_s}{\lambda} - 1 \right) \int (Q_b + \varepsilon)_y (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B ((\varepsilon + Q_b)^5 - Q_b^5) \\
& + 2 \int \Phi_b (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B ((\varepsilon + Q_b)^5 - Q_b^5) \\
& + 2 \int \Psi_b (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B ((\varepsilon + Q_b)^5 - Q_b^5) \\
& = f_{1,1}^{(i)} + f_{1,2}^{(i)} + f_{1,3}^{(i)} + f_{1,4}^{(i)} + f_{1,5}^{(i)}.
\end{aligned}$$

Term $f_{1,1}^{(i)}$: This term contains the leading order negative quadratic terms thanks to our choice of orthogonality conditions and suitable repulsivity properties of the

virial quadratic form¹¹ on the soliton core, and intrinsic monotonicity properties of the renormalized (KdV) flow in the moving frame at speed 1 which expulses energy to the left and leads to positive terms induced by localization of both mass and energy.

Let us first integrate by parts in order to obtain a more manageable formula:

$$\begin{aligned} f_{1,1}^{(i)} &= 2 \int [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)]_y [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)] \psi_B \\ &\quad + 2 \int [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)]_y (-\psi'_B \varepsilon_y + \varepsilon(\varphi_{i,B} - \psi_B)). \end{aligned}$$

We compute the various terms separately:

$$\begin{aligned} &2 \int [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)]_y \psi_B [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)] \\ &= - \int \psi'_B [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)]^2 \\ &= - \int \psi'_B [-\varepsilon_{yy} + \varepsilon]^2 \\ &\quad - \int \psi'_B \left\{ [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)]^2 - [-\varepsilon_{yy} + \varepsilon]^2 \right\} \\ &= - \left[\int \psi'_B (\varepsilon_{yy}^2 + 2\varepsilon_y^2) + \int \varepsilon^2 (\psi'_B - \psi_B''') \right] \\ &\quad - \int \psi'_B \left\{ [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)]^2 - [-\varepsilon_{yy} + \varepsilon]^2 \right\}. \end{aligned}$$

Next after integration by parts:

$$\begin{aligned} &2 \int [-\varepsilon_{yy} + \varepsilon]_y [-\psi'_B \varepsilon_y + \varepsilon(\varphi_{i,B} - \psi_B)] \\ &= -2 \left\{ \int \psi'_B \varepsilon_{yy}^2 + \int \varepsilon_y^2 \left(\frac{3}{2} \varphi'_{i,B} - \frac{1}{2} \psi'_B - \frac{1}{2} \psi_B''' \right) \right. \\ &\quad \left. + \int \varepsilon^2 \left(\frac{1}{2} (\varphi_{i,B} - \psi_B)' - \frac{1}{2} (\varphi_{i,B} - \psi_B)''' \right) \right\}, \end{aligned}$$

similarly:

$$\begin{aligned} &- 2 \int [(Q_b + \varepsilon)^5 - Q_b^5]_y (\varphi_{i,B} - \psi_B) \varepsilon \\ &= - \frac{1}{3} \int (\varphi_{i,B} - \psi_B)' \{ [(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon - 6[(\varepsilon + Q_b)^5 - Q_b^5] \varepsilon] \\ &\quad - 2 \int (\varphi_{i,B} - \psi_B) (Q_b)_y [(Q_b + \varepsilon)^5 - Q_b^5 - 5Q_b^4 \varepsilon], \end{aligned}$$

and by direct expansion:

$$\int [(Q_b + \varepsilon)^5 - Q_b^5]_y \psi'_B \varepsilon_y = 5 \int \psi'_B \varepsilon_y \{ (Q_b)_y [(Q_b + \varepsilon)^4 - Q_b^4] + (Q_b + \varepsilon)^4 \varepsilon_y \}.$$

¹¹see Lemma 3.4

We collect the above computations and obtain the following

$$\begin{aligned}
f_{1,1}^{(i)} &= - \int [3\psi'_B \varepsilon_{yy}^2 + (3\varphi'_{i,B} + \psi'_B - \psi_B''') \varepsilon_y^2 + (\varphi'_{i,B} - \varphi_{i,B}''') \varepsilon^2] \\
&\quad - 2 \int \left[\frac{(\varepsilon + Q_b)^6}{6} - \frac{Q_b^6}{6} - Q_b^5 \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5) \varepsilon \right] (\varphi'_{i,B} - \psi'_B) \\
&\quad + 2 \int [(\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4 \varepsilon] (Q_b)_y (\psi_B - \varphi_{i,B}) \\
&\quad + 10 \int \psi'_B \varepsilon_y \{ (Q_b)_y [(Q_b + \varepsilon)^4 - Q_b^4] + (Q_b + \varepsilon)^4 \varepsilon_y \} \\
&\quad - \int \psi'_B \{ [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)]^2 - [-\varepsilon_{yy} + \varepsilon]^2 \} \\
&= (f_{1,1}^{(i)})^< + (f_{1,1}^{(i)})^\sim + (f_{1,1}^{(i)})^>
\end{aligned}$$

where $(f_{1,1}^{(i)})^{<,\sim,>}$ respectively corresponds to integration on $y < -\frac{B}{2}$, $|y| \leq \frac{B}{2}$, $y > \frac{B}{2}$.

For the region $y < -B/2$, we rely on monotonicity type arguments and estimate using (3.20):

$$\begin{aligned}
\int_{y < -B/2} \varepsilon^2 |\varphi_{i,B}''| &\lesssim \frac{1}{B^2} \int_{y < -B/2} \varepsilon^2 \varphi'_{i,B} \leq \frac{1}{100} \int_{y < -B/2} \varepsilon^2 \varphi'_{i,B}, \\
\int_{y < -B/2} \varepsilon_y^2 |\psi_B''| &\lesssim \frac{1}{B^2} \int_{y < -B/2} \varepsilon_y^2 \varphi'_{i,B} \leq \frac{1}{100} \int_{y < -B/2} \varepsilon_y^2 \varphi'_{i,B},
\end{aligned}$$

by choosing B large enough. Next, we recall the Sobolev bound¹²: $\forall B \geq 1$,

$$\begin{aligned}
\|\varepsilon^2 \sqrt{\varphi'_{i,B}}\|_{L^\infty(y < -\frac{B}{2})}^2 &\lesssim \|\varepsilon\|_{L^2}^2 \left(\int_{y < -\frac{B}{2}} \varepsilon_y^2 \varphi'_{i,B} + \int_{y < -\frac{B}{2}} \varepsilon^2 \frac{(\varphi_{i,B}'')^2}{\varphi'_{i,B}} \right) \\
&\lesssim \delta(\kappa^*) \int_{y < -B/2} (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}.
\end{aligned} \tag{3.25}$$

Remark 3.3. This estimate is linked to the L^2 critical nature of the problem and the smallness relies only on the global L^2 smallness (3.5) only, and requires no smallness of derivatives. It is the key to control the pure ε^6 non linear term in the functionals $\mathcal{F}_{i,j}$.

The homogeneity of the power nonlinearity then ensures (for B large and κ^* small):

$$\begin{aligned}
&\left| \int_{y < -B/2} \left[\frac{(\varepsilon + Q_b)^6}{6} - \frac{Q_b^6}{6} - Q_b^5 \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5) \varepsilon \right] (\varphi'_{i,B} - \psi'_B) \right| \\
&\lesssim \int_{y < -B/2} (\varepsilon^6 + |Q_b|^4 \varepsilon^2) \varphi'_{i,B} \lesssim \left(\delta(\kappa^*) + e^{-\frac{B}{10}} \right) \int_{y < -B/2} \varphi'_{i,B} (\varepsilon^2 + \varepsilon_y^2) \\
&\leq \frac{1}{100} \int_{y < -B/2} (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}
\end{aligned}$$

¹²see the proof of Lemma 6 in [24]

and similarly for κ^* small depending on B ,

$$\begin{aligned} & \left| \int_{y < -\frac{B}{2}} [(\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4 \varepsilon] (Q_b)_y (\psi_B - \varphi_{i,B}) \right| \\ & \lesssim B \int_{y < -\frac{B}{2}} (\varepsilon^2 |Q_b|^3 + |\varepsilon|^5) (|Q_y| + |b| |(P\chi_b)'|) \varphi'_{i,B} \\ & \leq \frac{1}{100} \int_{y < -B/2} (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} + b^4. \end{aligned}$$

We further estimate using (3.25) and $(\varphi'_i)^2 \lesssim \psi' \lesssim (\varphi'_i)^2$ for $y < -\frac{1}{2}$:

$$\begin{aligned} & \left| \int_{y < -\frac{B}{2}} \psi'_B \varepsilon_y \{ (Q_b)_y [(Q_b + \varepsilon)^4 - Q_b^4] + (Q_b + \varepsilon)^4 \varepsilon_y \} \right| \\ & \lesssim e^{-\frac{1}{2}B} \int_{y < -\frac{B}{2}} \varphi'_{i,B} (\varepsilon_y^2 + \varepsilon^2) + \int \psi'_B |\varepsilon|^4 |\varepsilon_y|^2 \\ & \leq \frac{1}{100} \int \varepsilon_{yy}^2 \psi'_B + \frac{1}{100} \int_{y < -B/2} (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}. \end{aligned}$$

Note that for the term $\int \psi'_B |\varepsilon|^4 |\varepsilon_y|^2$, we have proceeded as follows:

$$\begin{aligned} \int \psi'_B \varepsilon_y^2 \varepsilon^4 & \lesssim \|\varepsilon^2 (\psi'_B)^{\frac{1}{4}}\|_{L^\infty}^2 \int \varepsilon_y^2 (\psi'_B)^{\frac{1}{2}} \\ & \lesssim \|\varepsilon\|_{L^2}^2 \left(\int (\varepsilon_y^2 + \varepsilon^2) (\psi'_B)^{\frac{1}{2}} \right) \int \varepsilon_y^2 (\psi'_B)^{\frac{1}{2}} \\ & \lesssim \delta(\alpha^*) \left(\int \varepsilon_y^2 (\psi'_B)^{\frac{1}{2}} \right)^2 + \delta(\alpha^*) \int \varepsilon_y^2 \varphi'_{i,B} \end{aligned}$$

and

$$\begin{aligned} \left(\int \varepsilon_y^2 (\psi'_B)^{\frac{1}{2}} \right)^2 & = \left(- \int \varepsilon \varepsilon_{yy} (\psi'_B)^{\frac{1}{2}} + \frac{1}{2} \int \varepsilon^2 ((\psi'_B)^{\frac{1}{2}})'' \right)^2 \\ & \lesssim \left(\int \varepsilon^2 \right) \int (\varepsilon_{yy}^2 + \varepsilon^2) \psi'_B. \end{aligned}$$

Thus,

$$\int \psi'_B \varepsilon_y^2 \varepsilon^4 \lesssim \delta(\alpha^*) \int (\varepsilon_{yy}^2 + \varepsilon^2) \psi'_B + \delta(\alpha^*) \int \varepsilon_y^2 \varphi'_{i,B}.$$

The remaining nonlinear term is estimated using the local H^2 control provided by localization:

$$\begin{aligned} & \left| \int_{y < -\frac{B}{2}} \psi'_B \left\{ [-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)]^2 - [-\varepsilon_{yy} + \varepsilon]^2 \right\} \right| \\ & = \left| \int_{y < -\frac{B}{2}} \psi'_B (-2\varepsilon_{yy} + 2\varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)) ((\varepsilon + Q_b)^5 - Q_b^5) \right| \\ & \lesssim \frac{1}{100} \int_{y < -\frac{B}{2}} \psi'_B (|\varepsilon_{yy}|^2 + |\varepsilon|^2) + 100 \int_{y < -\frac{B}{2}} (\varphi'_{i,B})^2 (|\varepsilon| |Q_b|^4 + |\varepsilon|^5)^2 \\ & \lesssim \frac{1}{100} \int_{y < -B/2} [\varepsilon_{yy}^2 \psi'_B + (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}]. \end{aligned}$$

In the region $y > \frac{B}{2}$, $\psi_B(y) = 1$. We rely on (3.20) to estimate:

$$\int_{y>B/2} \varepsilon^2 |\varphi_{i,B}''| \lesssim \frac{1}{B^2} \int_{y>B/2} \varepsilon^2 \varphi_{i,B}' \leq \frac{1}{100} \int_{y>B/2} \varepsilon^2 \varphi_{i,B}',$$

and we use the exponential localization of Q_b to the right and the Sobolev bound

$$\|\varepsilon\|_{L^\infty(y>0)} \lesssim \|\varepsilon\|_{H^1(y>0)} \lesssim \mathcal{N}_2^{\frac{1}{2}} \lesssim \delta(\kappa^*)$$

to control:

$$\begin{aligned} & \left| \int_{y>B/2} \left(\frac{(\varepsilon + Q_b)^6}{6} - \frac{Q_b^6}{6} - Q_b^5 \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5) \varepsilon \right) \varphi_{i,B}' \right| \\ & \lesssim \int_{y>B/2} (\varepsilon^6 + |Q_b|^4 \varepsilon^2) \varphi_{i,B}' \lesssim (\delta(\kappa^*) + e^{-\frac{B}{10}}) \int_{y>B/2} \varphi_{i,B}' (\varepsilon^2 + \varepsilon_y^2) \\ & \leq \frac{1}{100} \int_{y>B/2} (\varepsilon_y^2 + \varepsilon^2) \varphi_{i,B}', \\ & \left| \int_{y>B/2} [(\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4 \varepsilon] (Q_b)_y (\psi_B - \varphi_{i,B}) \right| \\ & \lesssim \int_{y>B/2} (\varepsilon^2 |Q_b|^3 + |\varepsilon|^5) (|Q_y| + |b|e^{-|y|}) \leq \frac{1}{100} \int_{y>B/2} (\varepsilon_y^2 + \varepsilon^2) \varphi_{i,B}'. \end{aligned}$$

In the region $|y| < B/2$, $\varphi_{i,B}(s, y) = 1 + y/B$ and $\psi_B(y) = 1$. In particular, $\varphi_{i,B}''' = \psi_B' = 0$ in this region, and we obtain:

$$\begin{aligned} (f_{1,1}^{(i)})^\sim &= -\frac{1}{B} \int_{|y|<B/2} \left\{ 3\varepsilon_y^2 + \varepsilon^2 \right. \\ & \quad \left. + 2 \left(\frac{(\varepsilon + Q_b)^6}{6} - \frac{Q_b^6}{6} - Q_b^5 \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5) \varepsilon \right) \right. \\ & \quad \left. + 2 ((\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4 \varepsilon) y (Q_b)_y \right\} \\ &= -\frac{1}{B} \int_{|y|<B/2} \left\{ 3\varepsilon_y^2 + \varepsilon^2 - 5Q_b^4 \varepsilon^2 + 20yQ'Q^3 \varepsilon^2 \right\} + R_{\text{Vir}}(\varepsilon), \end{aligned}$$

where

$$\begin{aligned} R_{\text{Vir}}(\varepsilon) &= -\frac{1}{B} \int_{|y|<B/2} \left\{ -5(Q_b^4 - Q^4) \varepsilon^2 + 20y((Q_b)_y Q_b^3 - Q'Q^3) \varepsilon^2 \right. \\ & \quad \left. - \frac{40}{3} Q_b^3 \varepsilon^3 - 15Q_b^2 \varepsilon^4 - 8Q_b \varepsilon^5 - \frac{5}{3} \varepsilon^6 \right. \\ & \quad \left. + 20y(Q_b)_y Q_b^2 \varepsilon^3 + 10y(Q_b)_y Q_b \varepsilon^4 + 2y(Q_b)_y \varepsilon^5 \right\}. \end{aligned}$$

We now claim the following coercivity result which is the main tool to measure dispersion (related to the Viriel estimate, see Section A.2).

Lemma 3.4 (Localized viriel estimate). *There exists $B_0 > 100$ and $\mu_3 > 0$ such that if $B \geq B_0$, then*

$$\int_{|y|<B/2} (3\varepsilon_y^2 + \varepsilon^2 - 5Q_b^4 \varepsilon^2 + 20yQ'Q^3 \varepsilon^2) \geq \mu_3 \int_{|y|<B/2} (\varepsilon_y^2 + \varepsilon^2) - \frac{1}{B} \int \varepsilon^2 e^{-\frac{|y|}{2}}.$$

We further estimate by Sobolev's inequality,

$$|R_{\text{Vir}}(\varepsilon)| \lesssim \frac{1}{B}(|b| + \|\varepsilon\|_{L^\infty(|y| < B/2)}) \int_{|y| < B/2} (\varepsilon_y^2 + \varepsilon^2) \lesssim \frac{1}{B} \delta(\kappa^*) \int_{|y| < B/2} (\varepsilon_y^2 + \varepsilon^2),$$

and thus for κ^* small enough:

$$(f_{1,1}^{(i)})^\sim \leq -\frac{\mu_3}{2B} \int_{|y| < B/2} (\varepsilon_y^2 + \varepsilon^2) + \frac{1}{B^2} \int \varepsilon^2 e^{-\frac{|y|}{2}}.$$

The collection of above estimates yields the bound:

$$f_{1,1}^{(i)} \leq -\frac{\mu_4}{B} \int [\psi'_B \varepsilon_{yy}^2 + \varphi'_{i,B}(\varepsilon_y^2 + \varepsilon^2)] + Cb^4 \quad (3.26)$$

for some universal $\mu_4 > 0$ independent of B .

Term $f_{1,2}^{(i)}$: We integrate by parts to express $f_{1,2}$:

$$\begin{aligned} f_{1,2}^{(i)} &= 2 \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q(L\varepsilon) - 2 \left(\frac{\lambda_s}{\lambda} + b \right) \int \varepsilon(1 - \varphi_{i,B}) \Lambda Q \\ &\quad + 2b \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda(\chi_b P) (-(\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B[(Q_b + \varepsilon)^5 - Q_b^5]) \\ &\quad + 2 \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q (-(\psi_B)_y \varepsilon_y - (1 - \psi_B) \varepsilon_{yy} + (1 - \psi_B)[(Q_b + \varepsilon)^5 - Q_b^5]) \\ &\quad + 2 \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q[(Q_b + \varepsilon)^5 - Q_b^5 - 5Q_b^4 \varepsilon] \end{aligned}$$

Observe from (2.20):

$$\int \Lambda Q(L\varepsilon) = (\varepsilon, L\Lambda Q) = -2(\varepsilon, Q) = 0.$$

We now use the orthogonality conditions $(\varepsilon, y\Lambda Q) = 0$ and the definition of $\varphi_{i,B}$ to estimate:

$$\left| \int \Lambda Q \varepsilon (1 - \varphi_{i,B}) \right| = \left| \int \Lambda Q \varepsilon \left(1 - \varphi_{i,B} + \frac{y}{B} \right) \right| \lesssim e^{-\frac{B}{8}} \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}},$$

so that by (2.29) and for B large enough:

$$\begin{aligned} \left| \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q \varepsilon (1 - \varphi_{i,B}) \right| &\lesssim \left(\mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} + b^2 \right) e^{-\frac{B}{8}} \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} \\ &\leq \frac{1}{500} \frac{\mu_4}{B} \mathcal{N}_{i,\text{loc}} + Cb^4. \end{aligned}$$

For the next term in $f_{1,2}^{(i)}$, we first integrate by parts to remove all derivatives on ε . Then, by (2.29), the weighted Sobolev bound (3.25) and the properties of $\varphi_{i,B}$, ψ_B , P and χ_b (2.9), we obtain for κ^* small,

$$\begin{aligned} &\left| 2b \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda(\chi_b P) (-(\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B[(Q_b + \varepsilon)^5 - Q_b^5]) \right| \\ &\lesssim |b| \left(\mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} + b^2 \right) \left(\int_{y < 0} e^{\frac{y}{B}} + 1 \right)^{\frac{1}{2}} \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} \\ &\lesssim |b| \left(\mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} + b^2 \right) B^{\frac{1}{2}} \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} \leq \frac{1}{500} \frac{\mu_4}{B} \mathcal{N}_{i,\text{loc}}(s) + Cb^4. \end{aligned}$$

Next, integrating by parts, using the exponential decay of Q and since $\psi_B(y) \equiv 1$ on $[-\frac{B}{2}, \infty)$:

$$\begin{aligned} & \left| \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q \left(-(\psi_B)_y \varepsilon_y - (1 - \psi_B) \varepsilon_{yy} + (1 - \psi_B) [(Q_b + \varepsilon)^5 - Q_b^5] \right) \right| \\ & \lesssim \left(\mathcal{N}_{i, \text{loc}}^{\frac{1}{2}} + b^2 \right) (e^{-\frac{B}{10}} + \delta(\kappa^*)) \mathcal{N}_{i, \text{loc}}^{\frac{1}{2}} \leq \frac{1}{500} \frac{\mu_4}{B} \mathcal{N}_{i, \text{loc}}, \end{aligned}$$

and finally:

$$\begin{aligned} & \left| \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q [(Q_b + \varepsilon)^5 - Q_b^5 - 5Q_b^4 \varepsilon] \right| \\ & \lesssim \left(\mathcal{N}_{i, \text{loc}}^{\frac{1}{2}} + b^2 \right) \delta(\kappa^*) \mathcal{N}_{i, \text{loc}}^{\frac{1}{2}} \leq \frac{1}{500} \frac{\mu_4}{B} \mathcal{N}_{i, \text{loc}}. \end{aligned}$$

The collection of above estimates yields the bound:

$$|f_{1,2}^{(i)}| \leq \frac{1}{100} \frac{\mu_4}{B} \mathcal{N}_{i, \text{loc}} + Cb^4.$$

Term $f_{1,3}^{(i)}$: We use the identity

$$\begin{aligned} & \int \psi_B(Q_b)_y ((\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4 \varepsilon) + \int \psi_B \varepsilon_y ((\varepsilon + Q_b)^5 - Q_b^5) \\ & = \frac{1}{6} \int \psi_B \partial_y [(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon] = -\frac{1}{6} \int \psi'_B [(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon] \end{aligned}$$

to compute:

$$\begin{aligned} f_{1,3}^{(i)} & = 2 \left(\frac{x_s}{\lambda} - 1 \right) \int \frac{1}{6} \psi'_B [(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon] \\ & + 2 \left(\frac{x_s}{\lambda} - 1 \right) \int (b\chi_b P + \varepsilon)_y [-\psi'_B \varepsilon_y - \psi_B \varepsilon_{yy} + \varepsilon \varphi_{i,B}] \\ & + 2 \left(\frac{x_s}{\lambda} - 1 \right) \int Q' [L\varepsilon - \psi'_B \varepsilon_y + (1 - \psi_B) \varepsilon_{yy} - \varepsilon(1 - \varphi_{i,B})] \\ & + 10 \left(\frac{x_s}{\lambda} - 1 \right) \int \varepsilon \psi_B (Q_b^4 (Q_b)_y - Q_b^4 Q_y). \end{aligned}$$

Since $|(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon| \lesssim |\varepsilon|^2 + |\varepsilon|^6$, by (3.25) and $|\frac{x_s}{\lambda} - 1| \leq \delta(\kappa^*)$, we have

$$\begin{aligned} & \left| 2 \left(\frac{x_s}{\lambda} - 1 \right) \int \frac{1}{6} \psi'_B [(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon] \right| \\ & \lesssim \delta(\kappa^*) \int \psi'_B (|\varepsilon|^2 + |\varepsilon|^6) \leq \frac{1}{500} \frac{\mu_4}{B} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}. \end{aligned}$$

Then, as before, integrating by parts, and using Cauchy-Schwarz inequality,

$$\begin{aligned} & \left| 2b \left(\frac{x_s}{\lambda} - 1 \right) \int (\chi_b P)_y [-\psi'_B \varepsilon_y - \psi_B \varepsilon_{yy} + \varepsilon \varphi_{i,B}] \right| \\ & \lesssim |b| \left(\mathcal{N}_{i, \text{loc}}^{\frac{1}{2}} + b^2 \right) B^{\frac{1}{2}} \mathcal{N}_{i, \text{loc}} \leq \frac{1}{500} \frac{\mu_4}{B} \mathcal{N}_{i, \text{loc}} + b^4. \end{aligned}$$

$$\begin{aligned} & \left| 2 \left(\frac{x_s}{\lambda} - 1 \right) \int \varepsilon_y [-\psi'_B \varepsilon_y - \psi_B \varepsilon_{yy} + \varepsilon \varphi_{i,B}] \right| \\ & \lesssim \delta(\kappa^*) \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} \leq \frac{1}{500} \frac{\mu_4}{B} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}. \end{aligned}$$

The next term is treated using the cancellation $LQ' = 0$ and the orthogonality conditions $(\varepsilon, \Lambda Q) = (\varepsilon, Q) = 0$, so that $(yQ', \varepsilon) = 0$. Thus, by the definitions of $\varphi_{i,B}$ and ψ_B ,

$$\begin{aligned} & \left| 2 \left(\frac{x_s}{\lambda} - 1 \right) \int Q' [L\varepsilon - \psi'_B \varepsilon_y + (1 - \psi_B) \varepsilon_{yy} - \varepsilon(1 - \varphi_{i,B})] \right| \\ &= \left| 2 \left(\frac{x_s}{\lambda} - 1 \right) \int Q' \left[-\psi'_B \varepsilon_y + (1 - \psi_B) \varepsilon_{yy} - \varepsilon \left(1 + \frac{y}{B} - \varphi_{i,B} \right) \right] \right| \\ &\lesssim \left(\mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} + b^2 \right) e^{-\frac{B}{10}} \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} \leq \frac{1}{500} \frac{\mu_4}{B} \mathcal{N}_{i,\text{loc}} + b^4. \end{aligned}$$

Finally,

$$\begin{aligned} & \left| 10 \left(\frac{x_s}{\lambda} - 1 \right) \int \varepsilon \psi_B (Q_b^4 (Q_b)_y - Q^4 Q_y) \right| \\ &\lesssim |b| \left(\mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} + b^2 \right) B^{\frac{1}{2}} \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} \leq \frac{1}{500} \frac{\mu_4}{B} \mathcal{N}_{i,\text{loc}} + Cb^4. \end{aligned}$$

In conclusion for $f_{1,3}^{(i)}$,

$$|f_{1,3}^{(i)}| \leq \frac{1}{100} \frac{\mu_4}{B} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} + Cb^4,$$

for B large enough and κ^* small enough.

Term $f_{1,4}^{(i)}$: We compute explicitly:

$$f_{1,4}^{(i)} = -2b_s \int (\chi_b + \gamma y(\chi_b)_y) P (-\psi_B \varepsilon_{yy} - \psi'_B \varepsilon_y + \varepsilon \varphi_{i,B} - \psi_B ((\varepsilon + Q_b)^5 - Q_b^5)).$$

We estimate after integrations by parts

$$\begin{aligned} \left| \int (\chi_b + \gamma y(\chi_b)_y) P (-\psi_B \varepsilon_y)_y \right| &\lesssim \int |\varepsilon| |(\psi_B ((\chi_b + \gamma y(\chi_b)_y) P)_y)_y| \\ &\lesssim B^{\frac{1}{2}} \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}}. \end{aligned}$$

$$\left| \int (\chi_b + \gamma y(\chi_b)_y) P \varepsilon \varphi_{i,B} \right| \lesssim B^{\frac{1}{2}} \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}}.$$

The estimate of the nonlinear term follows from the weighted Sobolev estimate (3.25) with $\psi \leq (\varphi'_i)^2$ for $y < -\frac{1}{2}$:

$$\begin{aligned} & \left| \int (\chi_b + \gamma y(\chi_b)_y) P \psi_B [(\varepsilon + Q_b)^5 - Q_b^5] \right| \lesssim \int \psi_B (|Q_b|^4 |\varepsilon| + |\varepsilon|^5) \\ &\lesssim B^{\frac{1}{2}} \left(\int (|\varepsilon|^2 + |\varepsilon|^6) \psi_B \right)^{\frac{1}{2}} \lesssim B^{\frac{1}{2}} \left(\int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} \right)^{\frac{1}{2}}. \end{aligned}$$

Together with (2.30), these estimates yield the bound:

$$|f_{1,4}| \leq \frac{1}{500} \frac{\mu_4}{B} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} + C|b|^4.$$

Term $f_{1,5}^{(i)}$: This term generates the leading order term in b through the error term Ψ_b in the construction of the approximate Q_b profile. Recall:

$$f_{1,5}^{(i)} = 2 \int \Psi_b (-\psi_B \varepsilon_y)_y + \varepsilon \varphi_{i,B} - \psi_B ((\varepsilon + Q_b)^5 - Q_b^5)).$$

We now rely on (2.14) to estimate by integration by parts and Cauchy-Schwarz's inequality,

$$\left| \int (\Psi_b)_y \psi_B \varepsilon_y \right| \lesssim B^{\frac{1}{2}} b^2 \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} \leq \frac{1}{500} \frac{\mu_4}{B} \mathcal{N}_{i,\text{loc}} + C|b|^4.$$

By (2.13), $|\Psi_b| \leq b^2 + |b|^{1+\gamma} \mathbf{1}_{[-2,-1]}(|b|^\gamma y)$ and so by the exponential decay of $\varphi_{i,B}$ in the left,

$$\left| \int \Psi_b \varphi_{i,B} \varepsilon \right| \lesssim \left(b^2 B^{\frac{1}{2}} + e^{-\frac{1}{2|b|^\gamma}} \right) |b|^{1+\gamma} \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}} \leq \frac{1}{500} \frac{\mu_4}{B} \mathcal{N}_{i,\text{loc}} + C|b|^4.$$

For the nonlinear term, similarly and using (3.25),

$$\left| \int \Psi_b \psi_B [(\varepsilon + Q_b)^5 - Q_b^5] \right| \leq \frac{1}{500} \frac{\mu_4}{B} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} + C|b|^4.$$

The collection of above estimates yields the bound:

$$|f_{1,5}^{(i)}| \leq \frac{1}{100} \frac{\mu_4}{B} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} + |b|^4.$$

step 4 $f_2^{(i,j)}$ term.

We integrate by parts using (3.24):

$$\begin{aligned} f_2^{(i,j)} &= 2\mathcal{J}_{i,j} \int \varepsilon \varphi_{i,B} \left[(-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)^5 + Q_b^5)_y \right. \\ &\quad \left. + \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda Q_b + \left(\frac{x_s}{\lambda} - 1 \right) (Q_b + \varepsilon)_y + \Phi_b + \Psi_b \right] \end{aligned}$$

We integrate by parts, estimate all terms like for $f_1^{(i)}$ and use (3.13) which implies

$$|\mathcal{J}_{i,j}| \lesssim \delta(\kappa^*)$$

to conclude:

$$|f_2^{(i,j)}| \lesssim \delta(\kappa^*) \left[\int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} + |b|^4 \right].$$

step 5 $f_3^{(i,j)}$ term.

Recall:

$$\begin{aligned} f_3^{(i,j)} &= 2 \frac{\lambda_s}{\lambda} \int \Lambda \varepsilon \left(-(\psi_B \varepsilon_y)_y + (1 + \mathcal{J}_{i,j}) \varepsilon \varphi_{i,B} - \psi_B [(\varepsilon + Q_b)^5 - Q_b^5] \right) \\ &\quad + (\mathcal{J}_{i,j})_s \int \varphi_{i,B} \varepsilon^2 - 2(j-1) \frac{\lambda_s}{\lambda} \mathcal{F}_{i,j}. \end{aligned}$$

We integrate by parts to compute:

$$\begin{aligned} \int \Lambda \varepsilon (\psi_B \varepsilon_y)_y &= - \int \varepsilon_y^2 \psi_B + \frac{1}{2} \int \varepsilon_y^2 y \psi'_B, \\ \int (\Lambda \varepsilon) \varepsilon \varphi_{i,B} &= - \frac{1}{2} \int \varepsilon^2 y \varphi'_{i,B}, \\ \int \Lambda \varepsilon \psi_B [(\varepsilon + Q_b)^5 - Q_b^5] &= \frac{1}{6} \int (2\psi_B - y \psi'_B) [(\varepsilon + Q_b)^6 - Q_b^6 - 6Q_b^5 \varepsilon] \\ &\quad - \int \psi_B \Lambda Q_b ((\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4 \varepsilon). \end{aligned}$$

Thus,

$$\begin{aligned}
f_3^{(i,j)} &= \frac{\lambda_s}{\lambda} \int [(2 - 2(j-1))\psi_B - y\psi'_B] \varepsilon_y^2 \\
&\quad - \frac{1}{3} \frac{\lambda_s}{\lambda} \int [(2 - 2(j-1))\psi_B - y\psi'_B] [(\varepsilon + Q_b)^6 - Q_b^6 - 6Q_b^5\varepsilon] \\
&\quad + 2 \frac{\lambda_s}{\lambda} \int \psi_B \Lambda Q_b ((\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4\varepsilon) \\
&\quad + (\mathcal{J}_{i,j})_s \int \varphi_{i,B} \varepsilon^2 - \frac{\lambda_s}{\lambda} (1 + \mathcal{J}_{i,j}) \int y\varphi'_{i,B} \varepsilon^2 - 2(j-1) \frac{\lambda_s}{\lambda} (1 + \mathcal{J}_{i,j}) \int \varphi_{i,B} \varepsilon^2 \\
&= \frac{\lambda_s}{\lambda} \int [2(2-j)\psi_B - y\psi'_B] \varepsilon_y^2 \\
&\quad - \frac{1}{3} \frac{\lambda_s}{\lambda} \int [2(2-j)\psi_B - y\psi'_B] [(\varepsilon + Q_b)^6 - Q_b^6 - 6Q_b^5\varepsilon] \\
&\quad + 2 \frac{\lambda_s}{\lambda} \int \psi_B \Lambda Q_b ((\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4\varepsilon) \\
&\quad + \frac{1}{i} \left[(\mathcal{J}_{i,j})_s - 2(j-1)(1 + \mathcal{J}_{i,j}) \frac{\lambda_s}{\lambda} \right] \int (i\varphi_{i,B} - y\varphi'_{i,B}) \varepsilon^2 \\
&\quad + \frac{1}{i} \left[(\mathcal{J}_{i,j})_s - (2(j-1) + i)(1 + \mathcal{J}_{i,j}) \frac{\lambda_s}{\lambda} \right] \int y\varphi'_{i,B} \varepsilon^2 \\
&= f_{3,1}^{(i,j)} + f_{3,2}^{(i,j)},
\end{aligned}$$

where

$$f_{3,2}^{(i,j)} = \frac{1}{i} \left[(\mathcal{J}_{i,j})_s - (2(j-1) + i)(1 + \mathcal{J}_{i,j}) \frac{\lambda_s}{\lambda} \right] \int y\varphi'_{i,B} \varepsilon^2.$$

We estimate all terms in the above expression using again the notation $(f_{3,k}^{(i,j)})_{<,\sim,>}$ corresponding to integration on $y < -\frac{B}{2}$, $|y| < \frac{B}{2}$, $y > \frac{B}{2}$. The middle term is easily estimated in brute force using (3.13), (2.33), (2.29) and the a priori bound (3.5), we get

$$|(f_3^{(i,j)})_{\sim}| \lesssim \delta(\kappa^*) \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}.$$

For $y < -B$, we use the exponential decay of $\psi_B, \varphi_{i,B}$ and (3.20) to estimate:

$$\begin{aligned}
&\int_{y < -\frac{B}{2}} (\psi_B + |y|\psi'_B + \varphi_{i,B}) (\varepsilon_y^2 + \varepsilon^2) + |y|\varphi'_{i,B} \varepsilon^2 \\
&\lesssim \int_{y < -\frac{B}{2}} \varepsilon_y^2 \varphi'_{i,B} + \int_{y < -\frac{B}{2}} |y|\varphi'_{i,B} \varepsilon^2 \\
&\lesssim \int \varepsilon_y^2 \varphi'_{i,B} + \left(\int_{y < -\frac{B}{2}} |y|^{100} e^{\frac{y}{B}} \varepsilon^2 \right)^{\frac{1}{100}} \left(\int_{y < -\frac{B}{2}} e^{\frac{y}{B}} \varepsilon^2 \right)^{\frac{99}{100}} \\
&\lesssim \int \varepsilon_y^2 \varphi'_{i,B} + \mathcal{N}_{i,\text{loc}}^{\frac{9}{10}},
\end{aligned}$$

where we have used $\int_{y < -\frac{B}{2}} |y|^{100} e^{\frac{y}{B}} \varepsilon^2 \leq \|\varepsilon\|_{L^2}^2 \leq \delta(\kappa^*)$.

Remark 3.5. We see in the above estimate why we need to impose a stronger exponential weight on ε_y than on ε at $-\infty$ in the definition of $\mathcal{F}_{i,j}$. Indeed, since

the global L^2 norm of ε_y is not controlled¹³, we cannot estimate $\int_{y<0} |y| \psi'_B \varepsilon_y^2$ as we did for $\int_{y<0} |y| \varphi'_{i,B} \varepsilon^2$.

Together with (2.29) and the weighted Sobolev bound (3.25), this yields the bound:

$$|(f_3^{(i,j)})^{<}| \lesssim (b + \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}}) \left(\int \varepsilon_y^2 \varphi'_{i,B} + \mathcal{N}_{i,\text{loc}}^{\frac{9}{10}} \right) \lesssim \delta(\kappa^*) \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} + b^4.$$

For $y > B$, we estimate in brute force using (3.20)

$$i\varphi_{i,B} - y\varphi'_{i,B} = 0 \quad \text{for } y > B,$$

and (3.25),

$$|(f_{3,1}^{(i,j)})^{>}| \lesssim (b + \mathcal{N}_{i,\text{loc}}^{\frac{1}{2}}) \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} \lesssim \delta(\kappa^*) \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}.$$

It only remains to estimate $(f_{3,2}^{(i,j)})^{>}$. It is a dangerous term which requires:

- the weighted bound (3.7) and in particular its consequences (3.14), (3.15) which are additional information necessary to close the estimates;
- the following cancellation manufactured in the definition (3.9) from (2.33), (3.13):

$$\begin{aligned} & \left| (\mathcal{J}_{i,j})_s - (2(j-1) + i)(1 + \mathcal{J}_{i,j}) \frac{\lambda_s}{\lambda} \right| \\ &= \frac{4(j-1) + 2i}{(1 - J_1)^{4(j-1)+2i+1}} \left| (J_1)_s - \frac{1}{2} \frac{\lambda_s}{\lambda} (1 - J_1) \right| \lesssim |b| + \mathcal{N}_{i,\text{loc}} \end{aligned} \quad (3.27)$$

Remark 3.6. Note that the gain in (3.27) with respect to (2.29) motivates the presence of the term $(1 + \mathcal{J}_{i,j})$ in (3.8).

The estimates (3.27), (3.14), (3.15) together with the bootstrap bounds (3.5), (3.6) and the control (3.23) imply:

$$\begin{aligned} |(f_{3,2}^{(i,j)})^{>}| &\lesssim (|b| + \mathcal{N}_{i,\text{loc}}) \left(1 + \frac{1}{\lambda^{\frac{10}{9}}} \right) \mathcal{N}_{i,\text{loc}}^{\frac{8}{9}} \\ &\lesssim |b| \left(1 + \delta(\kappa^*) |b|^{-\frac{5}{9}} \right) \mathcal{N}_{i,\text{loc}}^{\frac{8}{9}} + \mathcal{N}_{i,\text{loc}} \left(1 + \delta(\kappa^*) \mathcal{N}_{i,\text{loc}}^{-\frac{5}{9}} \right) \mathcal{N}_{i,\text{loc}}^{\frac{8}{9}} \\ &\lesssim \delta(\kappa^*) (\mathcal{N}_{i,\text{loc}} + |b|^4). \end{aligned}$$

The collection of above estimates yields the bound:

$$|f_3^{(i,j)}| \lesssim \delta(\kappa^*) \left(\int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B} + |b|^4 \right).$$

step 6 $f_4^{(i)}$ term.

First,

$$|(Q_b)_s| = |b_s P(\chi(|b|^\gamma y) + \gamma |b|^\gamma y \chi'(|b|^\gamma y))| \lesssim |b_s|.$$

We use the following Sobolev bound:

$$\|\varepsilon^2 \sqrt{\psi_B}\|_{L^\infty}^2 \lesssim \delta(\kappa^*) \int (\varepsilon_y^2 + \varepsilon^2) \psi_B \quad (3.28)$$

¹³because λ becomes large in the (Exit) regime.

to obtain

$$\int \psi_B |\varepsilon|^5 \lesssim \|\psi_B^{\frac{1}{2}} \varepsilon^2\|_{L^\infty}^{\frac{3}{2}} \int \psi_B^{\frac{1}{4}} \varepsilon^2 \lesssim \left(\int \varepsilon^2 \right)^{\frac{3}{4}} \int (\varepsilon_y^2 + \varepsilon^2) \psi_B \lesssim \delta(\kappa^*) \int (\varepsilon_y^2 + \varepsilon^2) \psi_B,$$

and thus from (2.30), $|Q_b| \leq C$ and (3.20),

$$\begin{aligned} |f_4^{(i)}| &\lesssim |b_s| \int \psi_B (\varepsilon^2 |Q_b|^3 + |\varepsilon|^5) \lesssim (b^2 + \mathcal{N}_{i,\text{loc}}) \int (\varepsilon_y^2 + \varepsilon^2) \psi_B \\ &\lesssim \delta(\kappa^*) \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{i,B}. \end{aligned}$$

step 7 Proof of (3.12).

First, we estimate from the homogeneity of the nonlinearity and the Sobolev bound (3.28)

$$\int \psi_B |(\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5| dy \lesssim \int \psi_B (|Q_b|^4 \varepsilon^2 + |\varepsilon|^6) \lesssim \delta(\kappa^*) \int (\varepsilon_y^2 + \varepsilon^2) \psi_B.$$

The upper bound follows immediately.

The lower bound follows from the structure (3.8) of $\mathcal{F}_{i,j}$ which is a localization of the linearized Hamiltonian close to Q . Indeed, we rewrite:

$$\begin{aligned} \mathcal{F}_{i,j} &= \int \psi_B \varepsilon_y^2 + \varphi_{i,B} \varepsilon^2 - 5 \int Q^4 \varepsilon^2 + \mathcal{J}_{i,j} \int \varphi_{i,B} \varepsilon^2 \\ &\quad - \frac{1}{3} \int \psi_B [(\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5 - 15Q_b^4 \varepsilon^2] dy - 5 \int \psi_B (Q_b^4 - Q^4) \varepsilon^2 dy. \end{aligned}$$

The small L^2 term is estimated from (3.9), (3.13):

$$|\mathcal{J}_{i,j}| \int \varphi_{i,B} \varepsilon^2 \lesssim \delta(\kappa^*) \int \varphi_{i,B} \varepsilon^2,$$

The non linear term is estimated using the homogeneity of the nonlinearity and the Sobolev bound (3.28):

$$\begin{aligned} &\int \psi_B |(\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5 - 15Q_b^4 \varepsilon^2| dy \\ &\lesssim \int \psi_B (|Q_b|^3 |\varepsilon|^3 + |\varepsilon|^6) \lesssim \delta(\kappa^*) \int (\varepsilon_y^2 + \varepsilon^2) \psi_B. \end{aligned}$$

The coercivity of the linearized energy (2.3) together with the choice of orthogonality conditions (2.20) and a standard localization argument¹⁴ now ensure the coercivity for B large enough:

$$\int \psi_B \varepsilon_y^2 + \varphi_{i,B} \varepsilon^2 - 5 \psi_B Q^4 \varepsilon^2 \geq \mu \mathcal{N}_i,$$

and the lower bound (3.12) follows.

This concludes the proof of Proposition 3.1. □

¹⁴see for example the Appendix of [17] for more details

3.2. Dynamical control of the tail. We now provide an elementary dynamical control of the L^2 tail on the right of the soliton which will allow us to close the bootstrap bound (H3) of Proposition 3.1 in the setting of Theorem 1.2. Let a smooth function

$$\varphi_{10}(y) = \begin{cases} 0 & \text{for } y \leq 0, \\ y^{10} & \text{for } y \geq 1. \end{cases}, \quad \varphi'_{10} \geq 0.$$

Lemma 3.7 (Dynamical control of the tail on the right). *Under the assumptions of Proposition 3.1, there holds:*

$$\frac{1}{\lambda^{10}} \frac{d}{ds} \left\{ \lambda^{10} \int \varphi_{10} \varepsilon^2 \right\} \lesssim \mathcal{N}_{1,\text{loc}} + b^2. \quad (3.29)$$

Proof of Lemma 3.7. We compute from (3.24):

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \int \varphi_{10} \varepsilon^2 &= \int \varepsilon_s \varepsilon \varphi_{10} = \int \varphi_{10} \varepsilon \left[\frac{\lambda_s}{\lambda} \Lambda \varepsilon + (-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)^5 + Q_b^5)_y \right. \\ &\quad \left. + \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda Q_b + \left(\frac{x_s}{\lambda} - 1 \right) (Q_b + \varepsilon)_y + \Phi_b + \Psi_b \right]. \end{aligned}$$

We integrate by parts the linear term and use $y\varphi'_{10} = 10\varphi_{10}$ for $y \geq 1$ and $\varphi'''_{10} \ll \varphi'_{10}$ for y large enough to derive the bound

$$\begin{aligned} &\int \varphi_{10} \varepsilon \left[\frac{\lambda_s}{\lambda} \Lambda \varepsilon + (-\varepsilon_{yy} + \varepsilon)_y \right] \\ &= -\frac{1}{2} \frac{\lambda_s}{\lambda} \int y \varphi'_{10} \varepsilon^2 - \frac{3}{2} \int \varphi'_{10} \varepsilon_y^2 - \frac{1}{2} \int \varphi'_{10} \varepsilon^2 + \frac{1}{2} \int \varphi'''_{10} \varepsilon^2 \\ &\leq -\frac{10}{2} \frac{\lambda_s}{\lambda} \int \varphi_{10} \varepsilon^2 - \frac{1}{4} \int \varphi'_{10} (\varepsilon_y^2 + \varepsilon^2) + C \mathcal{N}_{1,\text{loc}}. \end{aligned}$$

The terms involving the geometrical parameters are controlled from the exponential localization of Q_b on the right and (2.29), (2.30):

$$\begin{aligned} \left| \frac{\lambda_s}{\lambda} + b \right| \left| \int \varphi_{10} \varepsilon (\Lambda Q_b) \right| &\lesssim (b + \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}}) \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}} \lesssim \mathcal{N}_{1,\text{loc}} + b^2, \\ \left| \frac{x_s}{\lambda} - 1 \right| \left| \int \varphi_{10} \varepsilon (Q_b + \varepsilon)_y \right| &\lesssim (b + \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}}) \left[\mathcal{N}_{1,\text{loc}}^{\frac{1}{2}} + \int \varphi'_{10} \varepsilon^2 \right] \\ &\lesssim \mathcal{N}_{1,\text{loc}} + b^2 + \delta(\kappa^*) \int \varphi'_{10} \varepsilon^2, \end{aligned}$$

$$\int |\varphi_{10} \varepsilon \Phi_b| \lesssim |b_s| \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}} \lesssim b^2 + \mathcal{N}_{1,\text{loc}}.$$

We control similarly the interaction with the error from (2.12):

$$\int |\varphi_{10} \varepsilon \Psi_b| \lesssim b^2 \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}} \lesssim b^2 + \mathcal{N}_{1,\text{loc}}.$$

By integration by parts in the nonlinear term, we can remove all derivatives on ε to obtain (using $|Q_b| + |(Q_b)_y| \leq C e^{-\frac{1}{2}y}$ for $y > 0$)

$$\begin{aligned} \left| \int \varphi_{10} \varepsilon [(\varepsilon + Q_b)^5 - Q_b^5]_y \right| &\lesssim \int_{y>0} \varphi_{10} e^{-\frac{1}{2}y} \varepsilon^2 (|\varepsilon|^3 + 1) + \int \varphi'_{10} \varepsilon^6 \\ &\lesssim \int_{y>0} e^{-\frac{1}{4}y} \varepsilon^2 (|\varepsilon|^3 + 1) + \int \varphi'_{10} \varepsilon^6 \end{aligned}$$

Thus, by standard Sobolev estimates,

$$\left| \int \varphi_{10} \varepsilon [(\varepsilon + Q_b)^5 - Q_b^5]_y \right| \lesssim \mathcal{N}_{1,\text{loc}} + \delta(\kappa^*) \int \varphi'_{10} (\varepsilon_y^2 + \varepsilon^2).$$

The collection of above estimates yields the bound:

$$\frac{d}{ds} \int \varphi_{10} \varepsilon^2 + 10 \frac{\lambda_s}{\lambda} \int \varphi_{10} \varepsilon^2 \lesssim \mathcal{N}_{1,\text{loc}} + b^2,$$

and (3.29) is proved. \square

4. Rigidity near the soliton. Proof of Theorem 1.2

This section is devoted to the proof of the following proposition which classifies the behavior of any solution close to Q and directly implies Theorem 1.2. Let $u_0 \in H^1$ with

$$u_0 = Q + \varepsilon_0, \quad \|\varepsilon_0\|_{H^1} < \alpha_0, \quad \int_{y>0} y^{10} \varepsilon_0^2(y) dy < 1, \quad (4.1)$$

and let $u(t)$ be the corresponding solution of (1.1) on $[0, T)$. Let \mathcal{T}_{α^*} be the L^2 modulated tube around the manifold of solitary waves given by (1.13) and define the exit time:

$$t^* = \sup\{0 < t < T, \text{ such that } \forall t' \in [0, t], u(t') \in \mathcal{T}_{\alpha^*}\}$$

which satisfies $t^* > 0$ by assumption on the data. We claim:

Proposition 4.1 (Rigidity-Dynamical version). *There exist universal constants $0 < \alpha_0^* \ll \alpha^* \ll \kappa^*$ and $C^* > 1$ such that the following holds. Let u_0 satisfy (4.1) with $0 < \alpha_0 < \alpha_0^*$, then $u(t)$ satisfies the assumptions (H1)-(H2)-(H3) of Proposition 3.1 on $[0, t^*)$.*

Moreover, let t_1^* be the separation time defined as:

$$\begin{aligned} t_1^* &= 0, \text{ if } |b(0)| \geq C^* \mathcal{N}_1(0), \\ t_1^* &= \sup\{0 < t < t^* \text{ such that } \forall t' \in [0, t], |b(t')| < C^* \mathcal{N}_1(t')\}, \text{ otherwise.} \end{aligned} \quad (4.2)$$

Then the following dichotomy holds:

(Soliton) If $t_1^* = t^*$ then $t_1^* = t^* = T = +\infty$. In addition,

$$\mathcal{N}_2(t) \rightarrow 0, \quad b(t) \rightarrow 0, \quad \text{as } t \rightarrow +\infty, \quad (4.3)$$

$$\lambda(t) = \lambda_\infty(1 + o(1)), \quad x(t) = \frac{t}{\lambda_\infty^2}(1 + o(1)), \quad \text{as } t \rightarrow +\infty, \quad (4.4)$$

for some λ_∞ satisfying $|\lambda_\infty - 1| \leq \delta(\alpha_0)$.

(Exit) If $t_1^* < t^*$ with $b(t_1^*) \leq -C^* \mathcal{N}_1(t_1^*)$, then $t^* < T$. In particular,

$$\inf_{\lambda_0 > 0, x_0 \in \mathbb{R}} \left\| u(t^*) - \frac{1}{\lambda_0^{\frac{1}{2}}} Q \left(\frac{\cdot - x_0}{\lambda_0} \right) \right\|_{L^2} = \alpha^*. \quad (4.5)$$

In addition:

$$\lambda(t^*) \geq \frac{C(\alpha^*)}{\delta(\alpha_0)}. \quad (4.6)$$

(Blow up) If $t_1^* < t^*$ with $b(t_1^*) \geq C^* \mathcal{N}_1(t_1^*)$, then $t^* = T$. In addition $T < +\infty$ and there exists $0 < \ell_0 < \delta(\alpha_0)$ such that

$$\lim_{t \rightarrow T} \frac{\lambda(t)}{(T-t)} = \ell_0, \quad \lim_{t \rightarrow T} \frac{b(t)}{(T-t)^2} = \ell_0^3, \quad \lim_{t \rightarrow T} (T-t)x(t) = \frac{1}{\ell_0^2}, \quad (4.7)$$

and there holds the bounds:

$$\|\varepsilon_x(t)\|_{L^2} \lesssim \lambda^2(t) [|E_0| + \delta(\alpha_0)], \quad \|\varepsilon(t)\|_{L^2} \lesssim \delta(\alpha_0). \quad (4.8)$$

Remark 4.2. Note that $u(t)$ belongs to the tube \mathcal{T}_{α^*} as long as $\frac{1}{3} \leq \lambda(t) \leq 3$ and that the three cases are equivalently characterized by:

(Soliton) For all t , $\lambda(t) \in [\frac{1}{2}, 2]$.

(Exit) There exists $t_0 > 0$ such that $\lambda(t_0) > 2$.

(Blow up) There exists $t_0 > 0$ such that $\lambda(t_0) < \frac{1}{2}$.

A continuity argument thus ensures that the cases (Exit) and (Blow up) are open in \mathcal{A} .

Also, note that on (t_1^*, t^*) , $\lambda(t)$ is almost monotonic known for $t > t_1^*$ and the separation time t_1^* defines a trapped regime i.e

$$|b(t)| \gtrsim C^* \mathcal{N}_1(t) \quad \text{for } t \geq t_1^*,$$

and hence the scenario is chosen at this point.

The rest of this section is devoted to the proof of Proposition 4.1. First, note that by Lemma 2.5, u admits a decomposition on $[0, t^*]$:

$$u(t, x) = \frac{1}{\lambda^{\frac{1}{2}}(t)} (Q_{b(t)} + \varepsilon) \left(t, \frac{x - x(t)}{\lambda(t)} \right)$$

with thanks to (4.1):

$$\|\varepsilon(0)\|_{H^1} + |b(0)| + |1 - \lambda(0)| \lesssim \delta(\alpha_0), \quad \int_{y>0} y^{10} \varepsilon^2(0) dy \leq 2. \quad (4.9)$$

In particular, arguing as in the proof of (3.14), we have

$$\mathcal{N}_2(0) \lesssim \delta(\alpha_0). \quad (4.10)$$

For κ^* as in Proposition 3.1, define

$$t^{**} = \sup\{0 < t < t^* \text{ such that } u \text{ satisfies (H1)–(H2)–(H3) on } [0, t]\}.$$

Note that $t^{**} > 0$ is well-defined from (4.9), (4.10) and a straightforward continuity argument. Recall that $s = s(t)$ is the rescaled time (2.22), and we let $s^{**} = s(t^{**})$ and $s^* = s(t^*)$. One important step of the proof is to obtain $t^{**} = t^*$ by improving (H1)–(H2)–(H3) on $[0, t^{**}]$.

4.1. Consequence of the monotonicity formula. We start with *coupling* the dispersive bounds (3.10), (3.11) with the modulation equation for b given by (2.37) to derive the key rigidity property at the heart of our analysis.

Lemma 4.3. *The following holds:*

1. Dispersive bounds. For $i = 1, 2$, for all $0 \leq s_1 \leq s_2 < s^{**}$,

$$\mathcal{N}_i(s_2) + \int_{s_1}^{s_2} \int (\varepsilon_y^2 + \varepsilon^2) (s) \varphi'_{i,B} ds \lesssim \mathcal{N}_i(s_1) + |b^3(s_2)| + |b^3(s_1)|, \quad (4.11)$$

$$\frac{\mathcal{N}_i(s_2)}{\lambda^2(s_2)} + \int_{s_1}^{s_2} \frac{\int (\varepsilon_y^2 + \varepsilon^2) (s) \varphi'_{i,B} + |b|^4}{\lambda^2(s)} ds \lesssim \frac{\mathcal{N}_i(s_1)}{\lambda^2(s_1)} + \left[\frac{|b^3(s_1)|}{\lambda^2(s_1)} + \frac{|b^3(s_2)|}{\lambda^2(s_2)} \right]. \quad (4.12)$$

2. Control of the dynamics for b . For all $0 \leq s_1 \leq s_2 < s^{**}$,

$$\int_{s_1}^{s_2} b^2(s) ds \lesssim \mathcal{N}_1(s_1) + |b(s_2)| + |b(s_1)|, \quad (4.13)$$

and for a universal constant $K_0 > 1$,

$$\left| \frac{b(s_2)}{\lambda^2(s_2)} - \frac{b(s_1)}{\lambda^2(s_1)} \right| \leq K_0 \left[\frac{b^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2)}{\lambda^2(s_2)} + \frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)} \right]. \quad (4.14)$$

3. Control of the scaling dynamics. Let $\lambda_0(s) = \lambda(s)(1 - J_1(s))^2$. Then on $[0, s^{**})$,

$$\left| \frac{(\lambda_0)_s}{\lambda_0} + b \right| \lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + |b| \left(\mathcal{N}_2^{\frac{1}{2}} + |b| \right). \quad (4.15)$$

Proof. *Proof of (4.11) and (4.12).* We first observe from (2.42) the bound:

$$b^2 \leq -b_s + C\mathcal{N}_{1,\text{loc}}. \quad (4.16)$$

By the monotonicity formula (3.10) with (3.12):

$$\begin{aligned} \mathcal{N}_i(s_2) + \int_{s_1}^{s_2} \int (\varepsilon_y^2 + \varepsilon^2) (s) \varphi'_{i,B} ds &\lesssim \mathcal{F}_{i,1}(s_2) + \mu \int_{s_1}^{s_2} \int (\varepsilon_y^2 + \varepsilon^2) (s) \varphi'_{i,B} ds \\ &\leq \mathcal{F}_{i,1}(s_1) + \int_{s_1}^{s_2} b^4(s) ds \\ &\lesssim \mathcal{N}_i(s_1) + \int_{s_1}^{s_2} b^4(s) ds \end{aligned}$$

and thus using (4.16), (3.4) and $|b|$ small,

$$\mathcal{N}_i(s_2) + \int_{s_1}^{s_2} \int (\varepsilon_y^2 + \varepsilon^2) (s) \varphi'_{i,B} ds \lesssim \mathcal{N}_i(s_1) + |b^3(s_2)| + |b^3(s_1)|.$$

Similarly, from (3.11), (3.12):

$$\begin{aligned} \frac{\mathcal{N}_i(s_2)}{\lambda^2(s_2)} + \int_{s_1}^{s_2} \frac{1}{\lambda^2(s)} \int (\varepsilon_y^2 + \varepsilon^2) (s) \varphi'_{i,B} ds \\ \lesssim \frac{\mathcal{F}_{i,2}(s_2)}{\lambda^2(s_2)} + \mu \int_{s_1}^{s_2} \frac{1}{\lambda^2(s)} \int (\varepsilon_y^2 + \varepsilon^2) (s) \varphi'_{i,B} ds \\ \lesssim \frac{\mathcal{F}_{i,2}(s_1)}{\lambda^2(s_1)} + \int_{s_1}^{s_2} \frac{b^4(s)}{\lambda^2(s)} ds \lesssim \frac{\mathcal{N}_i(s_1)}{\lambda^2(s_1)} + \int_{s_1}^{s_2} \frac{b^4(s)}{\lambda^2(s)} ds. \end{aligned} \quad (4.17)$$

We now integrate by parts in time using (4.16), (2.29) to estimate:

$$\begin{aligned} \int_{s_1}^{s_2} \frac{b^4(s)}{\lambda^2(s)} ds &\leq \int_{s_1}^{s_2} \frac{-b^2 b_s}{\lambda^2} + \delta(\kappa^*) \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}}(s)}{\lambda^2(s)} ds \\ &= -\frac{1}{3} \left[\frac{b^3}{\lambda^2} \right]_{s_1}^{s_2} - \frac{2}{3} \int_{s_1}^{s_2} b^3 \frac{\lambda_s}{\lambda^3} ds + \delta(\kappa^*) \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}}(s)}{\lambda^2(s)} ds \\ &\leq \left[\frac{|b^3(s_1)|}{\lambda^2(s_1)} + \frac{|b^3(s_2)|}{\lambda^2(s_2)} + \delta(\kappa^*) \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}}(s)}{\lambda^2(s)} ds \right] + \frac{2}{3} \int_{s_1}^{s_2} \frac{b^4(s)}{\lambda^2(s)} ds \\ &\quad + C \int_{s_1}^{s_2} \frac{|b|^3}{\lambda^2} \left[b^2 + \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}} \right] ds \\ &\leq \left[\frac{|b^3(s_1)|}{\lambda^2(s_1)} + \frac{|b^3(s_2)|}{\lambda^2(s_2)} \right] + \delta(\kappa^*) \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}}(s)}{\lambda^2(s)} ds + \left[\frac{2}{3} + \delta(\kappa^*) \right] \int_{s_1}^{s_2} \frac{b^4(s)}{\lambda^2(s)} ds \end{aligned}$$

and thus for κ^* small,

$$\int_{s_1}^{s_2} \frac{b^4(s)}{\lambda^2(s)} ds \lesssim \left[\frac{|b^3(s_1)|}{\lambda^2(s_1)} + \frac{|b^3(s_2)|}{\lambda^2(s_2)} \right] + \delta(\kappa^*) \int_{s_1}^{s_2} \frac{\mathcal{N}_{1,\text{loc}}(s)}{\lambda^2(s)} ds. \quad (4.18)$$

Injecting this bound into (4.17) concludes the proof of (4.12).

The virtue of (4.11), (4.12) is to reduce the control of the full problem to the sole control of the parameter b which is driven by the sharp ODE (2.37).

Proof of (4.13) and (4.14). The estimate (4.13) is derived by integrating (4.16) in time using (4.11). We then compute from (2.37), (2.29) and the a priori bound¹⁵ $|J| \lesssim \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}}$:

$$\begin{aligned} \left| \frac{d}{ds} \left\{ \frac{b}{\lambda^2} e^J \right\} \right| &= \left| \frac{d}{ds} \left\{ \frac{b}{\lambda^2} \right\} + \frac{b}{\lambda^2} J_s \right| e^J \lesssim \left| \frac{\lambda_s}{\lambda} \frac{b}{\lambda^2} J \right| + \frac{1}{\lambda^2} \left(\int \varepsilon^2 e^{-\frac{|y|}{10}} + |b|^3 \right) \\ &\lesssim \frac{b^2}{\lambda^2} |J| + \frac{1}{\lambda^2} (\mathcal{N}_{1,\text{loc}} + |b|^3) \lesssim \frac{b^2}{\lambda^2} \mathcal{N}_{1,\text{loc}}^{\frac{1}{2}} + \frac{1}{\lambda^2} (\mathcal{N}_{1,\text{loc}} + |b|^3) \\ &\lesssim \frac{1}{\lambda^2} (\mathcal{N}_{1,\text{loc}} + |b|^3). \end{aligned} \quad (4.19)$$

We integrate this estimate in time and use (4.16), (4.12) to get

$$\left| \left[\frac{b}{\lambda^2} e^J \right]_{s_1}^{s_2} \right| \lesssim \frac{\mathcal{N}_1(s_1)}{\lambda^2(s_1)} + \left[\frac{b^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2)}{\lambda^2(s_2)} \right]$$

and (4.14) follows from $|e^J - 1| \leq |J| \lesssim \mathcal{N}_1^{\frac{1}{2}}$ and (4.12).

Proof of (4.15). We integrate the scaling law using the sharp modulation equation (2.33). From (3.13):

$$\left| \frac{\lambda}{\lambda_0} - 1 \right| \lesssim |J_1| \lesssim \delta(\kappa^*), \quad (4.20)$$

and thus from (2.33), we get

$$\begin{aligned} \left| \frac{(\lambda_0)_s}{\lambda_0} + b - c_1 b^2 \right| &= \left| \frac{1}{1 - J_1} \left[(1 - J_1) \frac{\lambda_s}{\lambda} + b - 2(J_1)_s \right] - \frac{J_1}{1 - J_1} b \right| \\ &\lesssim \int \varepsilon^2 e^{-\frac{|y|}{10}} + |b|(\mathcal{N}_2 + |b|^2). \end{aligned}$$

This concludes the proof of Lemma 4.3. \square

We are now in position to prove the dichotomy of Proposition 4.1. Let

$$C^* = 10K_0 \quad (4.21)$$

where K_0 is the universal constant in (4.14) and let the separation time t_1^* be given by (4.2).

4.2. The soliton case. Assume that

$$t_1^* = t^* \quad \text{i.e. for all } t \in [0, t^*], |b(t)| \leq C^* \mathcal{N}_1(t). \quad (4.22)$$

We first prove that in this case $t^{**} = t^*$ which means that the bootstrap estimates (H1)–(H2)–(H3) of Proposition 3.1 hold on $[0, t^*]$. Indeed, we claim: $\forall s \in [0, s^{**})$,

$$|b(s)| + \mathcal{N}_2(s) + \|\varepsilon(s)\|_{L^2} + |1 - \lambda(s)| \lesssim \delta(\alpha), \quad (4.23)$$

$$\frac{|b(s)| + \mathcal{N}_2(s)}{\lambda^2(s)} \lesssim \delta(\alpha), \quad (4.24)$$

$$\int_{y>0} y^{10} \varepsilon^2(s) dy \leq 5. \quad (4.25)$$

¹⁵recall that J given by (2.36) is a well localized L^2 scalar product.

Taking $\alpha^* > 0$ small enough (compared to κ^*), this guarantees by a standard continuity argument that $t^{**} = t^*$.

Proof of (4.23)–(4.25). First, observe that by (3.22), (6.7) and the definition of t^{**} , on $[0, s^{**}]$,

$$\mathcal{N}_1 \lesssim \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{2,B}, \quad \mathcal{N}_1 \lesssim \mathcal{N}_2 \lesssim \delta(\kappa^*). \quad (4.26)$$

Therefore, from (4.22), (4.11) and (2.30): $\forall s \in [0, s^{**}]$,

$$\begin{aligned} |b(s) - b(0)| &\leq \int_0^s |b_s| ds \lesssim \int_0^s (b^2 + \mathcal{N}_{1,\text{loc}}) ds \lesssim \int_0^s (\delta(\kappa^*)(C^*)^2 + 1) \mathcal{N}_1(s) ds \\ &\lesssim \int_0^s \int (\varepsilon_y^2 + \varepsilon^2)(s) \varphi'_{2,B} ds \lesssim \mathcal{N}_2(0) + \delta(\kappa^*)(|b(s)| + |b(0)|). \end{aligned}$$

We thus conclude from (4.9) : $\forall s \in [0, s^{**}]$,

$$|b(s)| \lesssim |b(0)| + \mathcal{N}_2(0) \lesssim \delta(\alpha_0).$$

Then, from (4.11) and (4.13),

$$\mathcal{N}_2(s) + \int_0^s \left(b^2 + \int (\varepsilon_y^2 + \varepsilon^2)(s) \varphi'_{2,B} \right) ds \lesssim \delta(\alpha_0). \quad (4.27)$$

Injecting this into the conservation of the L^2 norm (2.27) using (2.15) ensures

$$\int |\varepsilon|^2 \lesssim \delta(\alpha_0),$$

and (4.23) is proved. Note that we also have from (3.13)

$$|J_1| + |J_2| \leq \delta(\alpha_0). \quad (4.28)$$

We now compute the variation of scaling from (4.15) which together with (4.22) implies:

$$\left| \frac{(\lambda_0)_s}{\lambda_0} \right| \lesssim |b| + \mathcal{N}_{1,\text{loc}} \lesssim \mathcal{N}_1 \lesssim \int (\varepsilon_y^2 + \varepsilon^2)(s) \varphi'_{2,B}$$

and thus from (4.27): $\forall 0 \leq s < s^{**}$,

$$\left| \log \left(\frac{\lambda_0(s)}{\lambda_0(0)} \right) \right| \lesssim \mathcal{N}_2(0) + \delta(\alpha_0) \lesssim \delta(\alpha_0).$$

Hence from (4.20), (4.28):

$$\left| \left(\frac{\lambda(s)}{\lambda(0)} - 1 \right) \right| \lesssim \delta(\alpha_0)$$

which with (4.9) implies:

$$\forall s \in [0, s^{**}], \quad |1 - \lambda(s)| \lesssim \delta(\alpha_0). \quad (4.29)$$

Together with (4.23), this implies (4.24). We now integrate (3.29) using (4.9), (4.29), (4.27) and obtain:

$$\begin{aligned} \int y^{10} \varepsilon^2(s) dy &\leq \frac{\lambda^{10}(0)}{\lambda^{10}(s)} \int y^{10} \varepsilon^2(0) dy + \frac{C}{\lambda^{10}(s)} \int_0^s \lambda^{10}(s) (\mathcal{N}_{1,\text{loc}} + b^2(s)) ds \\ &\leq 2 + \delta(\alpha_0) \leq 3 \end{aligned}$$

and (4.25) is proved.

We therefore conclude that $t^* = T$ and $u(t)$ remains in the tube \mathcal{T}_{α^*} for all $t \in [0, T)$ from (4.23). Moreover, inserting (4.23) in the conservation of the energy (2.28), we get

$$\forall t \in [0, T), \quad \|\varepsilon_y(t)\|_{H^1} \lesssim C.$$

Hence the solution $u(t)$ is uniformly bounded in H^1 and thus global: $T = +\infty$.

It remains to show the convergence (4.3)–(4.4). From (2.30), (4.27), (4.29):

$$\int_0^{+\infty} |b_t| dt \lesssim \int_0^{+\infty} |b_s| ds \lesssim \int_0^{+\infty} \left(b^2 + \int (\varepsilon_y^2 + \varepsilon^2)(s) \varphi'_{2,B} \right) ds \lesssim \delta(\alpha_0) \quad (4.30)$$

which implies

$$\lim_{t \rightarrow +\infty} b(t) = 0 \quad (4.31)$$

and the existence of a sequence $t_n \rightarrow \infty$ such that

$$\int (\varepsilon_y^2 + \varepsilon^2)(t_n) \varphi'_{2,B} \rightarrow 0 \quad \text{as } t_n \rightarrow +\infty.$$

By (4.26), $\mathcal{N}_1(t_n) \rightarrow 0$ as $n \rightarrow +\infty$ and thus using the monotonicity (4.11):

$$\mathcal{N}_1(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Together with the uniform bound (4.25), we also obtain

$$\mathcal{N}_2(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (4.32)$$

Finally, from (4.15), (4.30):

$$\int_0^{+\infty} \left| \log \left(\frac{(\lambda_0)_t}{\lambda_0} \right) \right| dt \lesssim \int_0^{+\infty} \left| \log \left(\frac{(\lambda_0)_s}{\lambda_0} \right) \right| ds \lesssim \delta(\alpha_0)$$

and thus

$$\lim_{t \rightarrow +\infty} \lambda_0(t) = \lambda_0^\infty \quad \text{with } |\lambda_0^\infty - 1| \lesssim \delta(\alpha_0).$$

Now from (4.32):

$$|J_1| \lesssim \mathcal{N}_2^{\frac{1}{2}} \rightarrow 0 \quad \text{as } t \rightarrow +\infty$$

and thus from (4.13):

$$\lim_{t \rightarrow +\infty} \lambda(t) = \lambda^\infty \quad \text{with } |\lambda^\infty - 1| \lesssim \delta(\alpha_0). \quad (4.33)$$

The translation parameter is controlled using (2.29) and (4.32), (4.33) which imply:

$$x_t = \frac{1}{\lambda^2} \frac{x_s}{\lambda} = \frac{1 + o(1)}{\lambda_\infty^2} \quad \text{as } t \rightarrow +\infty.$$

This concludes the proof of (4.3), (4.4).

4.3. Exit case. Now, we assume $t_1^* < t^*$ and

$$b(s_1^*) \leq -C^* \mathcal{N}_1(s_1^*). \quad (4.34)$$

Observe first that arguing on $[0, s_1^*]$ as in the soliton case, where the parameter b is controlled by \mathcal{N}_1 , we get $\forall s \in [0, s_1^*]$,

$$|\lambda(s) - 1| + |b(s)| + \mathcal{N}_2(s) + \int_0^s \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{2,B} ds \lesssim \delta(\alpha_0), \quad (4.35)$$

$$\int_{y>0} y^{10} \varepsilon^2(s) dy \leq 5. \quad (4.36)$$

In particular, $t_1^* < t^{**} \leq t^*$. Now, we claim

$$t^{**} = t^* \quad \text{and} \quad t^* < T,$$

which means that the solution leaves the tube $\mathcal{T}_{\alpha^*/2}$ in finite time.

*Proof of $t^{**} = t^*$.* We improve (H1)–(H2)–(H3) on $[t_1^*, t^{**}]$ to obtain $t^{**} = t^*$. The proof is different than the one for the soliton case since now b is not controlled by

\mathcal{N}_1 . The fundamental observation is that (4.14), (4.21), (4.34) immediately imply the rigidity:

$$\forall s \in [s_1^*, s^{**}), \quad -2\ell^* \leq \frac{b(s)}{\lambda^2(s)} \leq -\frac{\ell^*}{2} \quad (4.37)$$

where we have set from (4.35):

$$\ell^* = \frac{b(s_1^*)}{\lambda^2(s_1^*)} \leq -C^* \frac{\mathcal{N}_1(s_1^*)}{\lambda^2(s_1^*)} < 0, \quad |\ell^*| \lesssim \delta(\alpha_0). \quad (4.38)$$

Together with (4.12) and (4.35), this implies the bound:

$$\forall s \in [0, s^*], \quad \frac{|b(s)| + \mathcal{N}_2(s)}{\lambda^2(s)} \lesssim \delta(\alpha_0)$$

and (H2) is improved for α^* small compared to κ^* . We now observe using $b < 0$ from (4.37) and (4.15): $\forall s \in [s_1^*, s^{**})$,

$$\frac{(\lambda_0)_s(s)}{\lambda_0(s)} \gtrsim -\mathcal{N}_{1,\text{loc}}.$$

Together with (4.11) and the definition of λ_0 , this yields the almost monotonicity property of λ :

$$\forall s_1^* \leq \sigma_1 \leq \sigma_2 < s^{**}, \quad \lambda(\sigma_2) \geq \frac{1}{2}\lambda(\sigma_1). \quad (4.39)$$

We now integrate (3.29) using (4.11), (4.39), (4.35), (4.36) and (4.13) to obtain: $\forall s_1^* \leq s < s^{**}$,

$$\begin{aligned} & \int \varphi_{10} \varepsilon^2(s) dy \\ & \leq \frac{\lambda^{10}(s_1^*)}{\lambda^{10}(s)} \int \varphi_{10} \varepsilon^2(s_1^*) dy + \frac{C}{\lambda^{10}(s)} \int_{s_1^*}^s \lambda^{10}(s') (\mathcal{N}_{1,\text{loc}}(s') + b^2(s')) ds' \\ & \leq 3 + C \int_{s_1^*}^s (\mathcal{N}_{1,\text{loc}}(s') + b^2(s')) ds' \leq 3 + C(|b(s_1^*)| + |b(s)| + \mathcal{N}_1(s_1^*)) \\ & \leq 3 + \delta(\kappa^*) \end{aligned}$$

and (H3) is improved. We now improve (H1). Since $u(t) \in \mathcal{T}_{\alpha^*}$ on $[0, t^*)$, we have by (2.21), $\forall s \in [0, s^*)$, $|b(s)| \leq \delta(\alpha^*) \ll \kappa^*$. By (4.11), it follows that for all $s \in [0, s^{**})$, $N_2(s) \ll \kappa^*$. By (2.27), for all $s \in [0, s^{**})$ $\|\varepsilon(s)\|_{L^2} \ll \kappa^*$, and (H1) is improved. In conclusion, we have proved $t^{**} = t^*$ again in this case.

Proof of $t^ < T$.* Let us now show that (Exit) occurs in finite time. We divide (4.15) by λ_0^2 and use (4.37), (4.20) to estimate on $[t_1^*, t^*)$:

$$\frac{|\ell^*|}{3} - C \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} \leq (\lambda_0)_t \leq 3|\ell^*| + C \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2}.$$

Integrating in time, for all $t \in [t_1^*, t^*)$, we get

$$\frac{|\ell^*|(t - t_1^*)}{3} - C_1 \int_{t_1^*}^t \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} \leq \lambda_0(t) - \lambda_0(t_1^*) \leq 3|\ell^*|(t - t_1^*) + C_2 \int_{t_1^*}^{t_2} \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2}.$$

From the monotonicity (4.39) and then (4.11):

$$\int_{t_1^*}^t \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} = \int_{s_1^*}^s \lambda \mathcal{N}_{1,\text{loc}} \lesssim \lambda(s) \int_{s_1^*}^s \mathcal{N}_{1,\text{loc}} \lesssim \delta(\kappa^*) \lambda(t),$$

and we therefore obtain the bound: $\forall t \in [t_1^*, T^*)$,

$$\frac{1}{4} (|\ell^*|(t - t_1^*) + \lambda_0(t_1^*)) \leq \lambda(t) \leq 4 (|\ell^*|(t - t_1^*) + \lambda_0(t_1^*)).$$

This yields the following estimates on b from (4.37): $\forall t \in [t_1^*, t^*)$,

$$-40|\ell^*| (|\ell^*|(t - t_1^*) + \lambda_0(t_1^*))^2 \leq b(t) \leq -\frac{|\ell^*|}{40} (|\ell^*|(t - t_1^*) + \lambda_0(t_1^*))^2. \quad (4.40)$$

Injecting this bound into (4.11) yields the control

$$\mathcal{N}_2(t) \lesssim C(t)$$

which injected into the energy and mass conservation laws (2.27), (2.28) yields the H^1 bound

$$\|\varepsilon(t)\|_{H^1} \lesssim C(t).$$

It follows that $t^* = T < +\infty$ is not possible. On the other hand, $t^* = T = +\infty$ is also impossible since then by (4.40), $b(t) \rightarrow -\infty$ as $t \rightarrow +\infty$, which contradicts the definition of t^* . Thus, $t^* < T \leq +\infty$.

Finally, we observe that the scaling parameter is large at the exit time for α small compared to α^* . Indeed, $|b(t^*)| \gtrsim (\alpha^*)^4$ from (2.27) and thus from (4.37), (4.38):

$$\lambda^2(t^*) \geq \frac{1}{2} \frac{|b(t^*)|}{|\ell^*|} \geq \frac{C(\alpha^*)}{\delta(\alpha_0)}.$$

4.4. Blow up case. We now assume $t_1^* < t^*$ and

$$b(s_1^*) \geq C^* \mathcal{N}_1(s_1^*) > 0. \quad (4.41)$$

As before, we have $\forall s \in [0, s_1^*]$,

$$|\lambda(s) - 1| + |b(s)| + \mathcal{N}_2(s) + \int_0^s \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{2,B} ds \lesssim \delta(\alpha), \quad (4.42)$$

$$\int_{y>0} y^{10} \varepsilon^2(s) dy \leq 5. \quad (4.43)$$

In particular, $t_1^* < t^{**} \leq t^*$. In this case, we claim that $t^{**} = t^* = T$ and $T < \infty$.

*Proof of $t^{**} = t^* = T$.* First, we improve the bounds (H1)–(H2)–(H3) of Proposition 3.1. From (4.14), (4.21), (4.34), we recover the rigidity:

$$\forall s \in [s_1^*, s^{**}), \quad \frac{\ell^*}{2} \leq \frac{b(s)}{\lambda^2(s)} \leq 2\ell^* \quad (4.44)$$

where we set from (4.42):

$$\ell^* = \frac{b(s_1^*)}{\lambda^2(s_1^*)} > 0, \quad |\ell^*| \lesssim \delta(\alpha_0). \quad (4.45)$$

Together with (4.12) and (4.42), this implies the bound:

$$\forall s \in [0, s^*], \quad \frac{|b(s)| + \mathcal{N}_2(s)}{\lambda^2(s)} \lesssim \delta(\alpha_0)$$

and (H2) is improved provided α^* is small compared to κ^* . We now observe from $b > 0$ and (4.15): on $[s_1^*, s^{**})$,

$$\frac{(\lambda_0)_s}{\lambda_0(s)} \gtrsim -\mathcal{N}_{1,\text{loc}}$$

which together with (4.11) and the definition of λ_0 , yields the almost monotonicity:

$$\forall s_1^* \leq \sigma_1 \leq \sigma_2 < s^{**}, \quad \lambda(\sigma_2) \leq \frac{3}{2} \lambda(\sigma_1). \quad (4.46)$$

In particular, from (4.42):

$$\forall s \in [0, s^{**}), \quad \lambda(s) \leq 2. \quad (4.47)$$

This yields with (4.42), (4.44), (4.38), (4.11): for all $0 \leq s \leq s^{**}$,

$$|b(s)| \lesssim \lambda^2(s) \ell^* \lesssim \delta(\alpha_0), \quad \mathcal{N}_2(s) + \int_0^s \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{2,B} ds \lesssim \delta(\alpha_0).$$

The conservation of the L^2 norm (2.27) implies

$$\|\varepsilon\|_{L^2}^2 \lesssim \delta(\alpha_0) \quad (4.48)$$

and (H1) is improved. We now integrate (3.29) using (4.11), (4.47), (4.42), (4.13) and obtain: $\forall 0 \leq s < s^{**}$,

$$\begin{aligned} \int \varphi_{10} \varepsilon^2(s) dy &\leq \frac{\lambda^{10}(0)}{\lambda^{10}(s)} \int \varphi_{10} \varepsilon^2(0) dy + \frac{C}{\lambda^{10}(s)} \int_0^s \lambda^{10} (\mathcal{N}_{1,\text{loc}} + b^2) ds \\ &\leq \frac{1}{\lambda^{10}(s)} \left[5 + C \int_0^s (\mathcal{N}_{1,\text{loc}} + b^2) ds \right] \leq \frac{5 + \delta(\kappa^*)}{\lambda^{10}(s)}. \end{aligned}$$

and (H3) is improved. We conclude that $t^{**} = t^*$. Moreover, by (4.48), for α_0 small enough compared to α^* , we get $t^* = T$ since the condition in the definition of \mathcal{T}_{α^*} is also improved by this estimate.

Blow up in finite time. We now divide (4.15) by λ_0^2 and use (4.44), (4.20) to estimate on $[t_1^*, T)$:

$$\frac{|\ell^*|}{3} - C \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} \leq -(\lambda_0)_t \leq 3|\ell^*| + C \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2}.$$

We integrate in time and obtain in particular: for all $t \in [t_1^*, T)$,

$$0 \leq \lambda_0(t) \leq \lambda_0(t_1^*) - \frac{|\ell^*|(t - t_1^*)}{3} + C_1 \int_{t_1^*}^t \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2}. \quad (4.49)$$

Now from the bound (4.47) again and (4.11):

$$\int_{t_1^*}^t \frac{\mathcal{N}_{1,\text{loc}}}{\lambda^2} d\tau = \int_{s_1^*}^s \lambda(\sigma) \mathcal{N}_{1,\text{loc}} d\sigma \lesssim 2 \int_{s_1^*}^s \mathcal{N}_{1,\text{loc}} d\sigma \lesssim 1,$$

and thus (4.49) implies:

$$T < +\infty \quad \text{and in particular} \quad \lambda(t) \rightarrow 0 \text{ as } t \rightarrow T.$$

The conservation of energy (2.28) implies

$$\|\varepsilon_y(t)\|_{L^2}^2 \lesssim \lambda^2(t) |E_0| + \mathcal{N}_2(t) \quad (4.50)$$

and thus from (H2):

$$\|\varepsilon_y(t)\|_{L^2} + b(t) + \mathcal{N}_2(t) \rightarrow 0 \text{ as } t \rightarrow T. \quad (4.51)$$

Proof of (4.7)–(4.8). We estimate from (4.44), (4.19) and (4.12) using $T < +\infty$:

$$\int_0^{+\infty} \left| \frac{d}{ds} \left\{ \frac{b}{\lambda^2} e^J \right\} \right| ds \lesssim \int_0^\infty \frac{1}{\lambda^2} (\mathcal{N}_{1,\text{loc}} + |b|^3) ds < +\infty,$$

and thus $\frac{be^J}{\lambda^2}$ has a limit as $t \rightarrow T$. Moreover,

$$|J(t)| \lesssim \mathcal{N}_2^{\frac{1}{2}}(t) \rightarrow 0 \text{ as } t \rightarrow T$$

from (H2), and thus from (4.44), (4.45):

$$\frac{b(t)}{\lambda^2(t)} \rightarrow \ell_0 > 0, \quad t \rightarrow T, \quad \text{with } |\ell_0| \lesssim \delta(\alpha_0). \quad (4.52)$$

The time integration of (4.15) using (4.52), (4.46), (4.11) yields:

$$\begin{aligned} \left| \lambda_0(t) - \int_t^T \frac{b}{\lambda^2} dt' \right| &\lesssim \int_t^T \frac{\lambda(\mathcal{N}_{1,\text{loc}} + o(b))}{\lambda^2} dt' \lesssim \int_s^{+\infty} \lambda \mathcal{N}_{1,\text{loc}} ds' + o(T-t) \\ &\lesssim o(T-t) + \lambda(s) \int_s^{+\infty} \mathcal{N}_{1,\text{loc}} ds' = o(|T-t| + \lambda(t)) \end{aligned}$$

and thus using (4.52) again:

$$\lim_{t \rightarrow T} \frac{\lambda_0(t)}{(T-t)} = \ell_0.$$

Moreover from (4.20):

$$\left| \frac{\lambda(t)}{\lambda_0(t)} - 1 \right| \lesssim |J_1(t)| \rightarrow 0 \quad \text{as } t \rightarrow T.$$

The control of the translation parameter follows from (2.29) and (H2) which yield:

$$x_t = \frac{1}{\lambda^2} \frac{x_s}{\lambda} = \frac{1}{\lambda^2} (1 + o(1))$$

and (4.7) follows. Finally, the L^2 bound in (4.8) follows from (4.48), and the rest of (4.8) follows from (H2) and the conservation of energy (2.28):

$$\|\varepsilon_y(t)\|_{L^2}^2 \lesssim \lambda^2(t)|E_0| + |b(t)| + \mathcal{N}_2(t) \lesssim (|E_0| + \delta(\alpha_0))\lambda^2(t).$$

This concludes the proof of Proposition 4.1.

5. Blow up for $E_0 \leq 0$

In this section, we let an initial data

$$u_0 \in \mathcal{A} \quad \text{with } E_0 \leq 0.$$

We moreover assume that u_0 is not a solitary wave up to symmetries. We claim that the corresponding solution $u(t)$ to gKdV blows up in finite time in the (Blow up) regime described by Proposition 4.1.

Let us first recall the following standard orbital stability statement which follows from the variational characterization of the ground state and a standard concentration compactness argument:

Lemma 5.1 (Orbital stability). *Let $\alpha > 0$ small enough and a function $v \in H^1$ such that*

$$\left| \int v^2 - \int Q^2 \right| \leq \alpha, \quad E(v) \leq \alpha \int v_x^2,$$

then there exist $(\lambda_v, x_v) \in \mathbb{R}_+^ \times \mathbb{R}$ such that*

$$\|Q - \epsilon_0 \lambda_v^{\frac{1}{2}} v(\lambda_v x + x_v)\|_{H^1} \leq \delta(\alpha), \quad \epsilon_0 \in \{-1, 1\}.$$

For $\alpha > 0$ small enough compared to α^* , it follows from the conservation of mass and energy that u remains in the tube \mathcal{T}_{α^*} on $[0, T)$. Therefore, only the case (Blowup) and (Soliton) can occur in Proposition 4.1. We argue by contradiction and assume that (Soliton) occurs.

Case $E_0 < 0$: This case is particularly simple to treat using the estimates of Proposition 4.1. Indeed, the conservation of energy (2.28) with $E_0 < 0$ together with the asymptotic stability statements (4.3), (4.4) imply:

$$\lambda^2(t)|E_0| + \int |\varepsilon_y|^2 \lesssim |b(t)| + \mathcal{N}_1(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

and thus

$$\lambda(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty$$

hence contradicts the dynamics of λ (4.4).

Case $E_0 = 0$: This case is substantially more subtle and in particular there is no obvious obstruction to the (Soliton) dynamics. In fact, the conservation of energy (2.28) yields with (4.3), (4.4):

$$\int |\varepsilon_y|^2 \lesssim |b(t)| + \mathcal{N}_1(t) \rightarrow 0 \quad \text{as } t \rightarrow +\infty, \quad (5.1)$$

but there is no further simple information on $\lambda(t)$. Our aim is to show that this \dot{H}^1 implies global L^2 dispersion, and hence the solution has minimal mass which for $E_0 = 0$ is possible only for the solitary wave itself.

By rescaling, we may without loss of generality assume that $\lambda_\infty = 1$ in (4.4). We claim

Lemma 5.2 (L^2 compactness). *Assume $E_0 = 0$ and $u(t)$ satisfies the (Soliton) case. Then*

$$\forall t \geq 0, \forall x_0 > 1, \quad \int_{x-x(t) < -x_0} u_x^2(t, x) dx \lesssim \frac{1}{x_0^3}, \quad (5.2)$$

$$\forall t \geq 0, \forall x_0 > 1, \quad \int_{x-x(t) < -x_0} u^2(t, x) dx \lesssim \frac{1}{\sqrt{x_0}}. \quad (5.3)$$

Assume Lemma 5.2, then from (4.3),

$$\forall x_0 > 1, \quad |b(t)| + \int_{y > -x_0} |\varepsilon(t, y)|^2 dy \rightarrow 0 \quad \text{as } t \rightarrow +\infty$$

and thus from (5.3), (4.4):

$$\int |u_0|^2 = \int u^2(t) = \int |Q_{b(t)} + \varepsilon(t)|^2 \rightarrow \int Q^2 \quad \text{as } t \rightarrow +\infty.$$

Hence u_0 has critical mass and a contradiction follows.

Proof of Lemma 5.2. Without loss of generality, by translation invariance, we assume that for all $t \geq 0$,

$$|\lambda(t) - 1| \leq \frac{1}{100}, \quad |x_t(t) - 1| \leq \frac{1}{100}, \quad \text{and} \quad \|\varepsilon(t)\|_{H^1} + |b(t)| \leq 1/100. \quad (5.4)$$

From the decomposition of $u(t)$, there exists $a_0 > 1$ such that, for α small enough, for all $t \in [0, T)$,

$$\int_{x < -\frac{1}{2}a_0} u^2(t, x + x(t)) dx \leq \int_{y < -\frac{1}{8}a_0} (\varepsilon(t) + Q_{b(t)})^2(y) dy \leq \frac{1}{100}. \quad (5.5)$$

Such $a_0 > 1$ is now fixed.

step 1 First decay property of u_x using almost monotonicity of a localized energy.

We claim that there exists $C > 0$ such that,

$$\forall t_0 \geq 0, \forall x_0 > a_0, \quad \int_{x-x(t_0) < -x_0} u_x^2(t_0, x) dx \leq \frac{C}{x_0^2}. \quad (5.6)$$

Proof of (5.6). Let ψ be a C^3 function such that for $c > 0$,

$$\psi \equiv 1 \text{ on } (-\infty, -3], \psi \equiv 0 \text{ on } [-\frac{1}{2}, +\infty), \quad (5.7)$$

$$\psi' = -\frac{1}{2} \text{ on } [-2, -1], \psi' \leq 0 \text{ on } \mathbb{R}, (\psi'')^2 \leq -c\psi', (\psi')^2 \leq c\psi \text{ on } \mathbb{R}.$$

Let $x_0 > a_0$. Define, for all $t > 0$,

$$E_{x_0}(t) = \int \left(u_x^2 - \frac{1}{3}u^6 \right) (t, x) \psi(\tilde{x}) dx, \quad (5.8)$$

where

$$\tilde{x} = \frac{x - x(t)}{\xi(t)}, \quad \xi(t) = x_0 + \frac{1}{4}(x(t) - x(t_0)).$$

First, observe that $\lim_{t \rightarrow +\infty} E_{x_0}(t) = 0$ by (5.1), (4.4) and the Gagliardo-Nirenberg inequality. Then, we control the variation of $E_{x_0}(t)$ on $[t_0, +\infty)$. By (2.50),

$$\begin{aligned} \frac{d}{dt} E_{x_0}(t) &= -\frac{1}{\xi(t)} \int (u_{xx} + u^5)^2 \psi'(\tilde{x}) - \frac{2}{\xi(t)} \int u_{xx}^2 \psi'(\tilde{x}) \\ &\quad + \frac{10}{\xi(t)} \int u^4 u_x^2 \psi'(\tilde{x}) + \frac{1}{\xi^3(t')} \int u_x^2 \psi'''(\tilde{x}) \\ &\quad - \frac{x_t(t)}{\xi(t)} \int \left(u_x^2 - \frac{1}{3}u^6 \right) \left(1 + \frac{1}{4}\tilde{x} \right) \psi'(\tilde{x}). \end{aligned} \quad (5.9)$$

All the integrals above are restricted to $\tilde{x} \in [-3, -\frac{1}{2}]$ since $\psi'(\tilde{x}) = 0$ for $\tilde{x} \notin [-3, -\frac{1}{2}]$. In particular, we have

$$-\frac{(x)_t(t)}{\xi(t)} \int u_x^2 \left(1 + \frac{1}{4}\tilde{x} \right) \psi'(\tilde{x}) \geq -\frac{1}{4} \frac{1}{\xi(t)} \int u_x^2 \psi'(\tilde{x}).$$

By (5.4) and $\|u\|_{L^\infty}^4 \lesssim \|u_x\|_{L^2}^2 \|u\|_{L^2}^2 \lesssim 1$,

$$\frac{10}{\xi(t)} \int u^4 u_x^2 |\psi'(\tilde{x})| \lesssim \frac{1}{\xi(t)} \|u\|_{L^\infty}^4 \int u_x^2 |\psi'(\tilde{x})| \leq \frac{1}{100} \frac{1}{\xi(t)} \int u_x^2 |\psi'(\tilde{x})|.$$

Moreover,

$$\left| \frac{1}{\xi^3(t)} \int u_x^2 \psi'''(\tilde{x}) \right| \lesssim \frac{1}{\xi^3(t)} \int u_x^2(t') \lesssim \frac{1}{\xi^3(t)}.$$

Now, we treat the u^6 term. Recall the following standard computation (see e.g. the proof of Lemma 6 in [24]), for a C^1 positive function ϕ such that $\frac{\phi'}{\sqrt{\phi}} \lesssim 1$, for all $v \in H^1(\mathbb{R})$,

$$\begin{aligned} \|v^2 \sqrt{\phi}\|_{L^\infty} &\leq \sup_{x \in \mathbb{R}} \left| \int_{-\infty}^x \left(2v'v \sqrt{\phi} + \frac{1}{2}v^2 \frac{\phi'}{\sqrt{\phi}} \right) \right| \\ &\lesssim \left(\int v^2 \right)^{\frac{1}{2}} \left(\int (v')^2 \phi + \int v^2 \frac{(\phi')^2}{\phi} \right)^{\frac{1}{2}}. \end{aligned} \quad (5.10)$$

Using this estimate, and the fact that $\frac{(\psi''(\tilde{x}))^2}{|\psi'(\tilde{x})|} \lesssim 1$, we obtain:

$$\begin{aligned} \|u^2 \sqrt{-\psi'(\tilde{x})}\|_{L^\infty}^2 &\lesssim \left(\int_{\text{supp } \phi} u^2 \right) \left(\int u_x^2 |\psi'(\tilde{x})| + \frac{1}{\xi^2(t)} \int u^2 \frac{(\psi''(\tilde{x}))^2}{|\psi'(\tilde{x})|} \right) \\ &\lesssim \left(\int_{\text{supp } \phi} u^2 \right) \left(\int u_x^2 |\psi'(\tilde{x})| + \frac{C}{\xi^2(t)} \int u^2 \right). \end{aligned} \quad (5.11)$$

Since $x_0 > a_0$, by (5.5), we have

$$\int_{\tilde{x} \in [-3, -\frac{1}{2}]} u^2(t) \leq \int_{x < -\frac{1}{2}x_0} u^2(t, x + x(t)) dx \leq \frac{1}{100}.$$

Thus, we get

$$\begin{aligned} \left| \int_{\tilde{x} \in [-3, -1/2]} u^6 \psi'(\tilde{x}) \right| &\lesssim \left(\int_{\tilde{x} \in [-3, -1/2]} u^2 \right)^2 \left(\int u_x^2 |\psi'(\tilde{x})| + \frac{C}{\xi^2} \int u^2 \right) \\ &\leq \frac{1}{100} \int u_x^2 |\psi'(\tilde{x})| + \frac{C}{\xi^2} \int u^2. \end{aligned} \quad (5.12)$$

Combining these estimates, we get

$$\frac{d}{dt} E_{x_0}(t) \gtrsim \frac{1}{\xi(t)} \int u_{xx}^2(t) |\psi'(\tilde{x})| + \frac{1}{\xi(t)} \int u_x^2 |\psi'(\tilde{x})| - C \xi^{-3}(t). \quad (5.13)$$

Integrating between t_0 and $+\infty$, using $\lim_{t \rightarrow +\infty} E_{x_0}(t) = 0$, and (5.4), we get

$$E_{x_0}(t_0) = \int (u_x^2 - \frac{1}{3} u^6)(t_0) \psi \left(\frac{x - x(t_0)}{x_0} \right) dx \lesssim \frac{1}{x_0^2}, \quad (5.14)$$

$$\int_0^{+\infty} \left[\int u_{xx}^2(t) |\psi'(\tilde{x})| + \int u_x^2 |\psi'(\tilde{x})| \right] \frac{dt}{\xi(t)} \lesssim \frac{1}{x_0^2}. \quad (5.15)$$

Using (5.10) and (5.5), we have

$$\begin{aligned} &\int u^6(t_0) \psi \left(\frac{x - x(t_0)}{x_0} \right) dx \\ &\lesssim \left(\int_{\tilde{x} \leq -\frac{1}{2}} u^2(t_0) \right)^2 \left(\int u_x^2(t_0) \psi \left(\frac{x - x(t_0)}{x_0} \right) + \frac{1}{x_0^2} \int u^2(t_0) \right) \\ &\leq \frac{1}{100} \int u_x^2(t_0) \psi \left(\frac{x - x(t_0)}{x_0} \right) + \frac{C}{x_0^2} \int u^2(t_0). \end{aligned}$$

Therefore, for all $t_0 \in [0, T)$, $x_0 > a_0$, we have obtained

$$\int_{x - x(t_0) < -x_0} u_x^2(t_0, x) + u^6(t_0, x) dx \lesssim \frac{1}{x_0^2}. \quad (5.16)$$

Since $\psi'(\tilde{x}) = 0$ for $\tilde{x} < -3$ and $\tilde{x} > -\frac{1}{2}$, using (5.15), we have

$$\int_0^{+\infty} \int u_{xx}^2(t') \psi(\tilde{x}) dt' < \infty.$$

Moreover,

$$\begin{aligned} \frac{d}{dx_0} \left(\int_0^{+\infty} \int u_{xx}^2(t) \psi(\tilde{x}) dt \right) &= \int_0^{+\infty} \int u_{xx}^2(t) \frac{-\tilde{x}}{\xi(t)} \psi'(\tilde{x}) dt \\ &\lesssim \int_0^{+\infty} \frac{1}{\xi(t)} \int u_{xx}^2(t) \psi'(\tilde{x}) dt \lesssim \frac{1}{x_0^2}. \end{aligned}$$

Integrating in x_0 , we get $\int_0^{+\infty} \int u_{xx}^2 \psi(\tilde{x}) dt' \leq \frac{C}{x_0}$ and arguing in a similar way for u_x , we obtain the following

$$\int_0^{+\infty} \int [u_{xx}^2(t) \psi(\tilde{x}) + u_x^2(t) \psi(\tilde{x})] dt \leq \frac{1}{x_0}. \quad (5.17)$$

step 2 Refined decay property of u_x .

We claim the improved decay:

$$\forall x_0 > 2a_0, \quad \int_{x < -x_0 + x(t_0)} u_x^2(t_0, x) dx \lesssim \frac{1}{x_0^3}. \quad (5.18)$$

To obtain this improved estimate, we introduce

$$G_{x_0}(t) = \int u_x^2(t) \psi(\tilde{x}).$$

By direct computations

$$\begin{aligned} \frac{d}{dt} G_{x_0}(t) &= -\frac{3}{\xi(t)} \int u_{xx}^2 \psi'(\tilde{x}) - \frac{x_t(t)}{\xi(t)} \int u_x^2 \left(1 + \frac{1}{4} \tilde{x}\right) \psi'(\tilde{x}) + \frac{1}{\xi^3(t)} \int u_x^2 \psi'''(\tilde{x}) \\ &\quad - 20 \int u_x^3 u^3 \psi(\tilde{x}) + \frac{5}{\xi(t)} \int u_x^2 u^4 \psi'(\tilde{x}). \end{aligned}$$

The second and the last terms in the right hand side are treated as before. For the third term, we use (5.16) and $\psi''' = 0$ for $\tilde{x} \geq -\frac{1}{2}$, which gives

$$\frac{1}{\xi^3(t)} \int u_x^2(t) \psi'''(\tilde{x}) \lesssim \frac{1}{\xi^3(t)} \int_{x \leq -\frac{1}{2}\xi(t)} u_x^2 \lesssim \frac{1}{\xi^5(t)} \lesssim \frac{\xi_t(t)}{\xi^5(t)}.$$

Finally, the term $\int u_x^3 u^3 \psi(\tilde{x})$ is controlled as follows, using (5.10) with $\phi = \psi(\tilde{x})$

$$\begin{aligned} \left| \int u_x^3 u^3 \psi(\tilde{x}) \right| &\leq \|u_x^2 \sqrt{\psi(\tilde{x})}\|_{L^\infty} \int |u_x u^3 \sqrt{\psi(\tilde{x})}| \\ &\leq \|u_x^2 \sqrt{\psi(\tilde{x})}\|_{L^\infty} \left(\int u_x^2 \psi(\tilde{x}) \right)^{\frac{1}{2}} \left(\int_{x < -\frac{1}{2}x_0 + x} u^6 \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_{x < -\frac{1}{2}x_0 + x} u_x^2 \right)^{\frac{1}{2}} \left(\left(\int u_{xx}^2 \psi(\tilde{x}) \right)^{\frac{1}{2}} + \left(\frac{1}{\xi^2} \int u_x^2 \psi(\tilde{x}) \right)^{\frac{1}{2}} \right) \\ &\quad \times \left(\int u_x^2 \psi(\tilde{x}) \right)^{\frac{1}{2}} \left(\int_{x < -\frac{1}{2}x_0 + x} u^6 \right)^{\frac{1}{2}} \\ &\lesssim \left(\int_{x < -\frac{1}{2}x_0 + x} u_x^2 \right)^{\frac{1}{2}} \left(\int_{x < -\frac{1}{2}x_0 + x} u^6 \right)^{\frac{1}{2}} \left(\int u_{xx}^2 \psi(\tilde{x}) + \int u_x^2 \psi(\tilde{x}) \right) \\ &\lesssim \frac{1}{x_0^2} \left(\int u_{xx}^2 \psi(\tilde{x}) + \int u_x^2 \psi(\tilde{x}) \right). \end{aligned}$$

In conclusion of these estimates, we have obtained

$$\frac{d}{dt} G_{x_0}(t) \gtrsim -\frac{\xi_t}{\xi^5} - \frac{1}{x_0^2} \left(\int (u_{xx}^2 + u_x^2) \psi(\tilde{x}) \right).$$

Therefore, by integration on $[t_0, +\infty)$, using (5.17) and $\lim_{t \rightarrow +\infty} G_{x_0}(t) = 0$, we obtain $G_{x_0}(t_0) \lesssim 1/x_0^3$, which proves (5.2).

step 3. L^2 estimate.

We deduce from (5.2) some L^2 tightness for u . Indeed, for $x_0 > 1$:

$$\begin{aligned} \|u(t, \cdot)\|_{L^\infty(x-x(t) < -x_0)}^2 &\lesssim \int_{x-x(t) < -x_0} |u_x u| dx \lesssim \left(\int_{x-x(t) < -x_0} u_x^2 \right)^{\frac{1}{2}} \left(\int u^2 \right)^{\frac{1}{2}} \\ &\lesssim \frac{1}{x_0^{3/2}} \end{aligned}$$

from which:

$$\int_{x-x(t) \leq -x_0} |u(t, x)|^2 dx \lesssim \int_{y > x_0} \frac{dy}{|y|^{\frac{3}{2}}} \lesssim \frac{1}{\sqrt{x_0}},$$

and (5.3) follows. \square

6. Sharp description of the blow up regime

We now finish the proof of Theorem 1.1 by proving (1.16) and (1.18) in the framework of a blow up solution in \mathcal{T}_{α^*} . We further use L^2 and H^1 monotonicity properties *away* from the soliton to propagate the dispersive information in larger regions to the left than the norm \mathcal{N}_i controlled by Proposition 4.1, and this will yield the sharp behavior (1.18).

We let:

$$\tilde{u}(t, x) = u(t, x) - \frac{1}{\lambda^{\frac{1}{2}}(t)} Q_{b(t)} \left(\frac{x - x(t)}{\lambda(t)} \right).$$

Proposition 6.1 (Improved dispersive bounds away from the soliton). *Let $u_0 \in \mathcal{A}$ such that $u(t)$ blows up in finite time T and:*

$$\forall t \in [0, T), \quad u(t) \in \mathcal{T}_{\alpha^*}.$$

Then, the following holds :

(i) H^1 estimates around the soliton:

$$\sup_{R > 1} \sup_{[T - \frac{1}{\ell_0^2 R}, T)} R^2 \int_{x > R} \tilde{u}^2(t, x) dx < \infty, \quad (6.1)$$

$$\lim_{R \rightarrow +\infty} \sup_{[T - \frac{1}{\ell_0^2 R}, T)} \int_{x > R} \tilde{u}_x^2(t, x) dx = 0, \quad (6.2)$$

$$\lim_{t \rightarrow T} \frac{1}{(T - t)^2} \int_{x - x(t) \geq -\frac{x(t)}{\log(T - t)}} \tilde{u}^2(t, x) dx = 0. \quad (6.3)$$

(ii) Existence and asymptotic of the dispersed remainder: *there exists $u^* \in H^1$ such that*

$$\tilde{u} \rightarrow u^* \quad \text{in } L^2 \quad \text{as } t \rightarrow T, \quad (6.4)$$

and

$$\int_{x > R} (u^*)^2(x) dx \sim \frac{\|Q\|_{L^1}^2}{8\ell_0 R^2} \quad \text{as } R \rightarrow +\infty. \quad (6.5)$$

The rest of this section is devoted to the proof of Proposition 6.1.

6.1. H^1 monotonicity away from the soliton. We aim at refining the dispersive estimate (4.12) by propagating it to the left of the solitary wave, since \mathcal{N}_2 involves an exponentially well localized norm at the left of the soliton. For this, we use H^1 monotonicity tools in the spirit of [24], [17].

Lemma 6.2 (Monotonicity away from the soliton core). *There exist $a_0 \ll 1$, $0 < \delta_0 \ll 1$ universal constants such that the following holds. Let $0 \leq t_0 < T$ close enough to T and $0 < \nu < \frac{1}{10}$ satisfying:*

$$\frac{\lambda^2(t_0)}{\nu} < \delta_0. \quad (6.6)$$

Let

$$\phi(x) = \frac{2}{\pi} \arctan \left(\exp \left(\frac{\sqrt{\nu}}{5} x \right) \right)$$

so that

$$\lim_{+\infty} \phi = 1, \quad \lim_{-\infty} \phi = 0, \quad \phi'''(x) \leq \frac{\nu}{25} \phi'(x), \quad |\phi''(x)| \lesssim \sqrt{\nu} \phi'(x), \quad \forall x \in \mathbb{R}. \quad (6.7)$$

Then: $\forall y_0 > a_0, \forall t_0 \leq t < T$, there holds the L^2 monotonicity bound:

$$\begin{aligned} & \int \tilde{u}^2(t, x) \phi \left(\frac{x - x(t_0)}{\lambda(t_0)} - \nu \frac{t - t_0}{\lambda^3(t_0)} + y_0 \right) dx + 2(b(t) - b(t_0))(P, Q) \\ & \lesssim \int \tilde{u}^2(t_0, x) \phi \left(\frac{x - x(t_0)}{\lambda(t_0)} + y_0 \right) dx + \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10} y_0} + \lambda^{2+\frac{1}{4}}(t_0) \end{aligned} \quad (6.8)$$

and the H^1 monotonicity bound:

$$\begin{aligned} & \int \left(\tilde{u}_x^2 - \frac{1}{3} \tilde{u}^6 \right) (t, x) \phi \left(\frac{5}{4} \left(\frac{x - x(t_0)}{\lambda(t_0)} - \nu \frac{t - t_0}{\lambda^3(t_0)} + y_0 \right) \right) dx \\ & - 2 \left(\frac{b(t)}{\lambda^2(t)} - \frac{b(t_0)}{\lambda^2(t_0)} \right) (P, Q) \\ & \lesssim \int \left(\tilde{u}_x^2(t_0, x) + \frac{\tilde{u}^2(t_0, x)}{\lambda^2(t_0)} \right) \phi \left(\frac{x - x(t_0)}{\lambda(t_0)} + y_0 \right) dx + \frac{1}{\sqrt{\nu}} \frac{e^{-\frac{\sqrt{\nu}}{10} y_0}}{\lambda^2(t_0)} + \lambda^{\frac{1}{4}}(t_0). \end{aligned} \quad (6.9)$$

The proof of Lemma 6.2 is postponed to Appendix A.

6.2. Proof of Proposition 6.1.

step 1 Proof of (6.1). The estimate (6.1) is a direct consequence of (6.8) and the space time control of local terms (4.12) which implies:

$$\frac{\mathcal{N}_i(t_2)}{\lambda^2(t_2)} + \int_{t_1}^{t_2} \frac{\int (\varepsilon_y^2 + \varepsilon^2)(s) \phi'_{i,B}}{\lambda^5(\tau)} d\tau \lesssim \delta(\alpha_0). \quad (6.10)$$

Indeed, fix $\nu = \frac{1}{16}$ in Lemma 6.2 (note that $B > 40 = 10/\sqrt{\sigma}$), then (6.6) is satisfied from the blow up assumption for t close enough to T , and we estimate the RHS of (6.8):

$$\begin{aligned} & \int \tilde{u}^2(t_0, x) \phi \left(\frac{x - x(t_0)}{\lambda(t_0)} + y_0 \right) dx = \int \varepsilon^2(t_0) \phi(y + y_0) \\ & \lesssim \int_{y < -y_0} \varepsilon^2(t_0) e^{\frac{\sqrt{\nu}}{10}(y+y_0)} + \int_{y > -y_0} \varepsilon^2(t_0) \\ & \lesssim \int_{y < -y_0} \varepsilon^2(t_0) e^{\frac{1}{B}(y+y_0)} + e^{\frac{y_0}{B}} \int_{-y_0 < y < 0} \varepsilon^2(t_0) e^{\frac{y}{B}} + \int_{y > 0} \varepsilon^2(t_0) \\ & \lesssim e^{\frac{y_0}{B}} \mathcal{N}_{1,\text{loc}}(t_0). \end{aligned} \quad (6.11)$$

Let then $R \gg 1$ large enough and t_R be such that $x(t_R) = R$, so that

$$T - t_R = \frac{1}{\ell_0^2 R} (1 + o_R(1)) = \frac{\lambda(t_R)}{\ell_0} (1 + o_R(1)) \quad \text{as } R \rightarrow +\infty. \quad (6.12)$$

We now make an essential use of the fact that the space time estimate (6.10) is better for local L^2 terms than the pointwise bound given by (H2). Indeed, the law (4.7) and (6.12) ensure, for R large:

$$\forall \tau \in [t_R - (R\ell_0)^{-\frac{5}{2}}, t_R], \quad \lambda(\tau) = \ell_0 [T - t_R + o_R(1)] \geq \frac{1}{2} \lambda(t_R)$$

and thus (6.10) implies:

$$(R\ell_0)^5 \int_{t_R - (R\ell_0)^{-\frac{5}{2}}}^{t_R} \int (\varepsilon_y^2 + \varepsilon^2)(t) \varphi'_{1,B} dt \lesssim \int_0^T \frac{\int (\varepsilon_y^2 + \varepsilon^2)(t) \varphi'_{1,B}}{\lambda^5(t)} dt \lesssim \delta(\alpha_0).$$

Thus, there exists $\bar{t}_R \in [t_R - (R\ell_0)^{-\frac{5}{2}}, t_R]$ such that

$$\int (\varepsilon_y^2 + \varepsilon^2)(\bar{t}_R) \varphi'_{1,B} \lesssim \delta(\alpha_0) (\ell_0 R)^{-\frac{5}{2}} \sim \delta(\alpha_0) (\lambda(\bar{t}_R))^{\frac{5}{2}} \quad (6.13)$$

which is a strict gain on the pointwise bound (H2). Note also the relations:

$$b(\bar{t}_R) = \ell_0^3 (T - \bar{t}_R)^2 (1 + o_R(1)) = \frac{1}{\ell_0 R^2} (1 + o_R(1)), \quad x(\bar{t}_R) = R + o_R(1). \quad (6.14)$$

We now apply (6.8) to $u(t)$ with

$$\nu = \frac{1}{16}, \quad t_0 = \bar{t}_R, \quad y_0 = y_R = 40 \log(\ell_0 R^3).$$

We obtain from (6.11), (6.13), (6.14) and $B \gg 1$: $\forall t \in [\bar{t}_R, T)$,

$$\begin{aligned} & \int \tilde{u}^2(t, x) \phi \left(\frac{x - x(\bar{t}_R)}{\lambda(\bar{t}_R)} - \frac{1}{16} \frac{t - \bar{t}_R}{\lambda^3(\bar{t}_R)} + y_R \right) - 2b(\bar{t}_R)(P, Q) \\ & \lesssim e^{\frac{y_R}{B}} \mathcal{N}_{1, \text{loc}}(\bar{t}_R) + e^{-\frac{1}{40} y_R} + (T - t_R)^{2+\frac{1}{4}} = o_R \left(\frac{1}{R^2} \right). \end{aligned} \quad (6.15)$$

and

$$2b(\bar{t}_R)(P, Q) = \frac{\|Q\|_{L^2}^2}{8\ell_0} \frac{1}{R^2} (1 + o_R(1)).$$

Moreover, we estimate using (6.14): $\forall x > 2R, \forall t \geq \bar{t}_R$:

$$\frac{x - x(\bar{t}_R)}{\lambda(\bar{t}_R)} - \frac{1}{16} \frac{t - \bar{t}_R}{\lambda^3(\bar{t}_R)} + y_R \geq \frac{2R - R}{\lambda(t_R)} - \frac{1}{16} \frac{t - \bar{t}_R}{\lambda^3(\bar{t}_R)} \geq \frac{1}{\ell_0 \lambda^2(t_R)} > 0.$$

Thus, from (6.14), (6.15), and also using $\phi(y) \geq \frac{1}{2}$ for $y > 0$, we obtain

$$\forall t \in \left[T - \frac{1}{2\ell_0^2 R}, T \right), \quad \int_{x > 2R} \tilde{u}^2(t, x) dx \lesssim \frac{1}{\ell_0 R^2},$$

and (6.1) follows.

step 2 Proof of (6.2). We now apply (6.9) to $u(t)$ with the same choice as before

$$\nu = \frac{1}{16}, \quad t_0 = \bar{t}_R, \quad y_0 = y_R = 40 \log(\ell_0 R^3).$$

We estimate like for the proof of (6.11) and using (6.13)

$$\int \tilde{u}_x^2(\bar{t}_R, x) \phi \left(\frac{x - x(\bar{t}_R)}{\lambda(\bar{t}_R)} + y_R \right) dx \lesssim e^{\frac{y_R}{B}} \frac{\int \varepsilon_y^2(\bar{t}_R) \varphi'_{1,B}}{\lambda^2(\bar{t}_R)} = o_R(1).$$

Using (6.11), we obtain for all $t \in [\bar{t}_R, T)$,

$$\begin{aligned} & \int \left(\tilde{u}_x^2(t, x) - \frac{1}{3} \tilde{u}^6(t, x) \right) \phi \left(\frac{5}{4} \left(\frac{x - x(\bar{t}_R)}{\lambda(\bar{t}_R)} - \frac{1}{16} \frac{t - \bar{t}_R}{\lambda^3(\bar{t}_R)} + y_R \right) \right) dx \\ & \lesssim \left| \frac{b(t)}{\lambda^2(t)} - \frac{b(\bar{t}_R)}{\lambda^2(\bar{t}_R)} \right| + R^2 e^{-\frac{1}{40} y_R} + o_R(1) \\ & = o_R(1) \end{aligned}$$

where we used $\lim_{t \rightarrow T} \frac{b(t)}{\lambda^2(t)} = \ell_0$ in the last step. Observe now the bound from Sobolev, (4.8) and (6.15):

$$\begin{aligned} & \int \tilde{u}^6(t, x) \phi \left(\frac{5}{4} \left(\frac{x - x(\bar{t}_R)}{\lambda(\bar{t}_R)} - \frac{1}{16} \frac{t - \bar{t}_R}{\lambda^3(\bar{t}_R)} + y_R \right) \right) dx \\ & \leq C \|\tilde{u}\|_{L^\infty}^4 \int \tilde{u}^2(t, x) \phi \left(\frac{x - x(\bar{t}_R)}{\lambda(\bar{t}_R)} - \frac{1}{16} \frac{t - \bar{t}_R}{\lambda^3(\bar{t}_R)} + y_R \right) dx \lesssim \frac{1}{R^2}, \end{aligned}$$

and (6.2) follows.

step 3 Proof of (6.3).

Let t be close to T . The space time estimate (6.10) and (4.7) ensure:

$$\frac{1}{\lambda^5(t)} \int_{t - \frac{20(T-t)}{|\log(T-t)|}}^{t - \frac{10(T-t)}{|\log(T-t)|}} \mathcal{N}_{1,\text{loc}}(\tau) d\tau \lesssim \int_0^T \frac{\mathcal{N}_{1,\text{loc}}(\tau)}{\lambda^5(\tau)} d\tau \lesssim \delta(\alpha),$$

and thus there exists

$$\bar{t} \in \left[t - \frac{20(T-t)}{|\log(T-t)|}, t - \frac{10(T-t)}{|\log(T-t)|} \right]$$

such that

$$\mathcal{N}_{1,\text{loc}}(\bar{t}) \leq \ell_0^5 (T-t)^4 |\log(T-t)|.$$

Moreover, from (4.7),

$$x(t) - x(\bar{t}) \geq (t - \bar{t}) \min_{[t, \bar{t}]} x_t \geq \frac{9}{\ell_0^2 (T-t) |\log(T-t)|} \geq \frac{8x(t)}{|\log(T-t)|}, \quad (6.16)$$

$$b(t) - b(\bar{t}) = o[(T-t)^2] \quad \text{as } t \rightarrow T.$$

We now apply (6.8) with:

$$\nu = \frac{1}{16}, \quad y_0 = \bar{y} = 40 |\log(T-t)|, \quad t_0 = \bar{t}.$$

The RHS of (6.8) is estimated using (6.11) and we obtain:

$$\int \tilde{u}^2(t, x) \phi \left(\frac{x - x(\bar{t})}{\lambda(\bar{t})} - \frac{1}{10} \frac{t - \bar{t}}{\lambda^3(\bar{t})} + \bar{y} \right) dx = o[(T-t)^2] \quad \text{as } t \rightarrow T.$$

Moreover, let x such that

$$x - x(t) \geq -\frac{x(t)}{|\log(T-t)|},$$

then from (6.16), (4.7),

$$\frac{x - x(\bar{t})}{\lambda(\bar{t})} - \frac{1}{10} \frac{t - \bar{t}}{\lambda^3(\bar{t})} \geq \frac{1}{\lambda(\bar{t}) |\log(T-t)|} \left[8x(t) - \frac{1}{10} \frac{10(T-t)}{\lambda^2(t)} \right] > 0,$$

and then $\phi(y) \geq \frac{1}{2}$ for $y > 0$ yields (6.3).

Remark 6.3. Observe that (6.1) and (6.2) imply:

$$\begin{aligned} \forall R > 1, \quad & \int_{x > R} \tilde{u}^2 \left(T - \frac{1}{200\ell_0^2 R}, x \right) dx \lesssim \frac{1}{R^2}, \\ & \lim_{R \rightarrow +\infty} \int_{x > R} \tilde{u}_x^2 \left(T - \frac{1}{200\ell_0^2 R}, x \right) dx = 0. \end{aligned}$$

In particular, given t close enough to T , we chose $R = (200\ell_0^2(T-t))^{-1} < \frac{1}{100}x(t)$ and conclude:

$$\int_{x > \frac{x(t)}{100}} \tilde{u}^2(t, x) dx \lesssim (T-t)^2, \quad \lim_{t \rightarrow T} \int_{x > \frac{x(t)}{100}} \tilde{u}_x^2(t, x) dx = 0. \quad (6.17)$$

step 4 L^2 tightness.

First observe from direct check using (4.7) that

$$\frac{1}{\lambda^{\frac{1}{2}}(t)} Q_{b(t)} \left(\frac{x - x(t)}{\lambda(t)} \right) - \frac{1}{\lambda^{\frac{1}{2}}(t)} Q \left(\frac{x - x(t)}{\lambda(t)} \right) \rightarrow 0 \quad \text{in } L^2 \quad \text{as } t \rightarrow T,$$

and hence (6.4) is equivalent to showing the existence of a strong limit

$$\tilde{u}(t) \rightarrow u^* \quad \text{in } L^2 \quad \text{as } t \rightarrow T. \quad (6.18)$$

We first claim that the sequence is tight: $\forall \epsilon > 0, \exists A_\epsilon > 1$ such that for all $t \in [0, T)$,

$$\int_{|x| > A(\epsilon)} \tilde{u}^2(t, x) dx < \epsilon. \quad (6.19)$$

On the right $x > A$ where non linear interactions take place, the claim directly follows from (6.1). On the left, this is a simple linear claim which follows from the finiteness of the time interval $[0, T)$, the H^1 bound (4.8) and a Kato L^2 localization argument. Indeed, let t_ϵ be close enough to T such that

$$\int_{t_\epsilon}^T \int_{x < 0} (u_x^2 + u^2) dx dt < \epsilon. \quad (6.20)$$

Let ψ be a C^3 function such that

$$\psi \equiv 1 \text{ on } (-\infty, -2], \quad \psi \equiv 0 \text{ on } [-1, +\infty), \quad \psi' \leq 0 \text{ on } \mathbb{R}. \quad (6.21)$$

Pick $A_\epsilon > 1$ large enough so that $\int u^2(t_\epsilon) \psi(x + A) \leq \epsilon$, then by (2.49),

$$\begin{aligned} & \frac{d}{dt} \int u^2(t) \psi(x + A) \\ &= -3 \int u_x^2(t) \psi'(x + A) + \int u^2(t) \psi'''(x + A) + \frac{5}{3} \int u^6(t) \psi'(x + A), \end{aligned}$$

and thus from (6.20): $\forall t \in [t_\epsilon, T)$,

$$\left| \int u^2(t) \psi(x + A) - \int u^2(t_\epsilon) \psi(x + A) \right| \leq C\epsilon.$$

and (6.19) follows. Now the uniform H^1 bound (4.8) ensures that for all sequence $t_n \rightarrow T$, there exists a subsequence $t_{\phi(n)} \rightarrow T$ and $u^* \in H^1$ such that $\tilde{u}(t_{\phi(n)}) \rightharpoonup u^*$ in H^1 weak and $\tilde{u}(t_{\phi(n)}) \rightarrow u^*$ L^2 strong from (6.19) and the local compactness of the Sobolev embedding. By a weak convergence argument, the limit u^* does not depend on the sequence (t_n) . Indeed, let θ be a C^∞ function with support in $[-K, K]$, then

$$\left| \frac{d}{dt} \int u \theta \right| = \left| \int u^5 \theta_x + \int u \theta_{xxx} \right| \leq C_\theta \int_{-K}^K (|u|^5 + |u|) \leq C_{\theta, K},$$

and thus $\int u(t) \theta$ has a limit as $t \rightarrow T$, and (6.18) follows. Note that the regularity $u^* \in H^1$ follows from (6.18), (4.8).

step 5 Universal behavior of u^* on the singularity.

We now turn to the proof of the universal behavior of u^* (6.5) on the singularity which follows from lower and upper bounds.

(i) *Upper bound:* Let $R \gg 1$ large enough. Let t_R be such that

$$x(t_R) = R,$$

so that from (4.7):

$$\begin{aligned} \frac{\lambda(t_R)}{\ell_0} &= (T - t_R)(1 + o_R(1)) = \frac{1}{\ell_0^2 R}(1 + o_R(1)), \\ b(t_R) &= \ell_0^3 (T - t_R)^2 (1 + o_R(1)) = \frac{1}{\ell_0 R^2}(1 + o_R(1)). \end{aligned}$$

We apply (6.8) to $u(t)$ with

$$\nu = \nu_R = \frac{1}{\log^2 R}, \quad y_0 = y_R = 10 \log^2(R^3), \quad t_0 = t_R$$

which satisfy the condition (6.6) for R large enough, and obtain: $\forall t \in [t_R, T)$,

$$\begin{aligned} &\int \tilde{u}^2(t, x) \phi \left(\frac{x - x(t_R)}{\lambda(t_R)} - \nu_R \frac{t - t_R}{\lambda^3(t_R)} + y_R \right) dx - 2b(t_R) \int PQ \\ &\lesssim \int \tilde{u}^2(t_R, x) \phi \left(\frac{x - x(t_R)}{\lambda(t_R)} + y_R \right) dx + \frac{1}{\nu_R} e^{-\frac{\sqrt{\nu_R}}{10} y_R} + (T - t_R)^{2+\frac{1}{4}} \\ &\lesssim \int \tilde{u}^2(t_R, x) \phi \left(\frac{x - x(t_R)}{\lambda(t_R)} + y_R \right) dx + o \left(\frac{1}{R^2} \right). \end{aligned}$$

Note that

$$\frac{-x(T_R)}{|\log(T - t_R)|} = -\frac{R}{\log R}(1 + o_R(1)) \ll \lambda(t_R) y_R$$

so that by (6.3) :

$$\begin{aligned} &\int \tilde{u}^2(t_R) \phi \left(\frac{x - x(t_R)}{\lambda(t_R)} + y_R \right) dx \\ &\lesssim e^{-\frac{\sqrt{\nu_R}}{10} y_R} \int \tilde{u}^2(t_R, x) dx + \int_{x - x(t_R) > -2\lambda(t_R) y_R} \tilde{u}^2(t_R, x) dx = \frac{1}{R^2} o_R(1). \end{aligned}$$

We thus conclude from (2.5):

$$\begin{aligned} &\int \tilde{u}^2(t, x) \phi \left(\frac{x - x(t_R)}{\lambda(t_R)} - \nu_R \frac{t - t_R}{\lambda^3(t_R)} + y_R \right) dx \\ &\leq \frac{2 \int PQ}{\ell_0 R^2} (1 + o_R(1)) = \frac{\|Q\|_{L^1}^2}{8 \ell_0 R^2} (1 + o_R(1)). \end{aligned}$$

Passing to the limit $t \rightarrow T$, we find

$$R^2 \int (u^*)^2(x) \phi \left(\frac{x - x(t_R)}{\lambda(t_R)} - \nu_R \frac{T - t_R}{\lambda^3(t_R)} + y_R \right) dx \leq \frac{\|Q\|_{L^1}^2}{8 \ell_0} (1 + o_R(1)).$$

Using $x(t_R) = R$ and $\lambda(t_R) = \frac{1}{R \ell_0} (1 + o_R(1))$, and passing to the limit $R \rightarrow +\infty$ yields:

$$\limsup_{R \rightarrow +\infty} R^2 \int_{x > (1 + \nu_R) R} (u^*)^2(x) dx \leq \frac{\|Q\|_{L^1}^2}{8 \ell_0},$$

which now easily implies:

$$\limsup_{R \rightarrow +\infty} R^2 \int_{x > R} (u^*)^2(x) dx \leq \frac{\|Q\|_{L^1}^2}{8 \ell_0}. \quad (6.22)$$

(ii) *Lower bound:* Let a smooth cut off function:

$$\omega \equiv 0 \text{ on } (-\infty, -1], \quad \omega \equiv 1 \text{ on } [0, +\infty), \quad \omega' \geq 0 \text{ on } \mathbb{R}.$$

Let $0 < \nu < \frac{1}{10}$ be arbitrary and let ω_ν be defined by $\omega_\nu(x) = \omega(\frac{x}{\nu})$. For $R > 1$ large, we define t_R such as $x(t_R) = R$ as before. Using the identity (2.49), we have, for all $t_R \leq t < T$,

$$\begin{aligned} & \frac{d}{dt} \int u^2 \omega_\nu \left(\frac{x - R + 4\log R}{R} \right) \\ & \geq -\frac{3}{R} \int u_x^2 \omega'_\nu \left(\frac{x - R + 4\log R}{R} \right) + \frac{1}{R} \int u^2 \omega'''_\nu \left(\frac{x - R + 4\log R}{R} \right) \\ & \geq -\frac{C_\nu}{R} \int_{(1-\nu)R < x < 4\log R < R} u_x^2 - \frac{C_\nu}{R^3} \int_{(1-\nu)R < x < 4\log R < R} u^2. \end{aligned}$$

By (6.2) and the properties of Q_b , (see in particular (2.9) and (2.11)), we have

$$\sup_{t \in [t_R, T]} \int_{(1-\nu)R < x < 4\log R < R} u_x^2(t, x) dx = o_R(1) \quad \text{as } R \rightarrow +\infty.$$

Since $T - t_R \lesssim \frac{1}{\ell_0^2 R}$, we obtain by integrating on $[t_R, t]$: $\forall t \in [t_R, T]$,

$$\int u^2(t) \omega_\nu \left(\frac{x - R + 4\log R}{R} \right) \geq \int u^2(t_R) \omega_\nu \left(\frac{x - R + 4\log R}{R} \right) + o_R \left(\frac{1}{R^2} \right). \quad (6.23)$$

We now develop u in terms of Q_b and \tilde{u} . On the one hand, a simple computation ensures:

$$\begin{aligned} & \int u^2(t) \omega_\nu \left(\frac{x - R + 4\log R}{R} \right) \\ & = \int Q^2 + \int \tilde{u}^2(t) \omega_\nu \left(\frac{x - R + 4\log R}{R} \right) + o_{t \rightarrow T}(1) \\ & \rightarrow \int Q^2 + \int (u^*)^2(t) \omega_\nu \left(\frac{x - R + 4\log R}{R} \right) \quad \text{as } t \rightarrow T. \end{aligned}$$

Next,

$$\begin{aligned} & \int u^2(t_R) \omega_\nu \left(\frac{x - R + 4\log R}{R} \right) \\ & = \int Q^2 + 2(P, Q)b(t_R) + \int \tilde{u}^2(t_R) \omega_\nu \left(\frac{x - R + 4\log R}{R} \right) + o_R \left(\frac{1}{R^2} \right) \\ & \geq \int Q^2 + \frac{1}{R^2} \left(\frac{\|Q\|_{L^1}^2}{8\ell_0} + o_R(1) \right) \end{aligned}$$

where we used (6.5) to treat the crossed term. We therefore conclude from (6.23):

$$\liminf_{R \rightarrow +\infty} R^2 \int_{x > (1-\nu)R - 4\log R} (u^*)^2(x) dx \geq \frac{\|Q\|_{L^1}^2}{8\ell_0}$$

and since ν is arbitrary,

$$\liminf_{R \rightarrow +\infty} R^2 \int_{x > R} (u^*)^2(x) dx \geq \frac{\|Q\|_{L^1}^2}{8\ell_0}.$$

This concludes the proof of Proposition 6.1.

Appendix A.

A.1. Proof of Lemma 6.2. Let $a_0 \gg 1$, $0 < \delta_0 \ll 1$ two constants to be chosen.

For $t_0 \in [0, T)$, we consider the renormalized solution

$$z(t', x') = \lambda^{\frac{1}{2}}(t_0) u(\lambda^3(t_0)t' + t_0, \lambda(t_0)x' + x(t_0)), \quad t' \in [0, T_z), \quad T_z = \frac{T - t_0}{\lambda^3(t_0)}. \quad (\text{A.1})$$

The function z admits a decomposition

$$\begin{aligned} z(t', x') &= \frac{1}{\lambda_z^{\frac{1}{2}}(t')} (Q_{b_z} + \varepsilon_z) \left(t', \frac{x - x_z(t')}{\lambda_z(t')} \right) \\ &= \frac{1}{\lambda_z^{\frac{1}{2}}(t')} Q_{b_z(t')} \left(\frac{x - x_z(t')}{\lambda_z(t')} \right) + \tilde{z}(t', x'), \end{aligned} \quad (\text{A.2})$$

with explicitly:

$$\begin{aligned} \varepsilon_z(t') &= \varepsilon(\lambda^3(t_0)t' + t_0), \quad \lambda_z(t') = \lambda(\lambda^3(t_0)t' + t_0)/\lambda(t_0), \\ x_z(t') &= (x(\lambda^3(t_0)t' + t_0) - x(t_0))/\lambda(t_0), \quad b_z(t') = b(\lambda^3(t_0)t' + t_0). \end{aligned}$$

In particular:

$$\lambda_z(0) = 1, \quad x_z(0) = 0, \quad b_z(0) = b(t_0). \quad (\text{A.3})$$

The monotonicity bound (4.46) and (4.8) ensure:

$$\forall t' \in [0, T_z), \quad \|(\varepsilon_z)_x(t')\|_{L^2}^2 \lesssim \lambda^2(t')(|E_0| + \delta(\alpha)), \quad \|\varepsilon_z(t')\|_{L^2}^2 \lesssim \delta(\alpha), \quad (\text{A.4})$$

$$\lambda_z(t') \leq \frac{3}{2}, \quad \|\tilde{z}(t')\|_{H^1} \lesssim \lambda^2(t_0)|E_0| + \delta(\alpha) \leq \delta_0 \quad (\text{A.5})$$

provided t_0 is close enough to T and α is small enough.

We denote by $\mathcal{N}_2(t')$ the quantity defined in (3.3) for $z(t')$. From (H2), and then (6.6), we have

$$\begin{aligned} \theta_z &= \sup_{t' \in [0, T_z]} \left| \frac{b_z(t') + \mathcal{N}_{2,z}(t')}{\lambda_z^2(t')} \right| = \sup_{t \in [t_0, T)} \lambda^2(t_0) \left| \frac{b(t) + \mathcal{N}_2(t)}{\lambda^2(t)} \right| \\ &\lesssim \lambda^2(t_0) \delta(\alpha) \lesssim \delta_0. \end{aligned} \quad (\text{A.6})$$

Lemma 6.2 follows directly from the following monotonicity result on \tilde{z} and (A.6).

Lemma A.1 (Monotonicity in renormalized variables). *Assume (A.3), (A.4), (A.5), (A.6), then $\forall y_0 > a_0$, $\forall t' \in [0, T_z)$, there holds:*

(i) L^2 monotonicity:

$$\begin{aligned} &\int \tilde{z}^2(t') \phi(x' - \nu t' + y_0) dx' + 2(P, Q)(b_z(t') - b_z(0)) \\ &+ \frac{1}{4} \int_0^{t'} \int (z_x^2 + \nu z^2)(t'') \phi'(x' - \nu t' + y_0) dx' dt'' \\ &\lesssim (\theta_z)^{\frac{9}{8}} + \int \tilde{z}^2(0) \phi(x' + y_0) dx' + \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10} y_0}. \end{aligned} \quad (\text{A.7})$$

(ii) H^1 monotonicity:

$$\begin{aligned} &\int \left[\tilde{z}_x^2 - \frac{1}{3} \tilde{z}^6 \right] (t') \phi \left(\frac{5}{4} (x' - \nu t' + y_0) \right) dx' - 2(P, Q) \left[\frac{b_z(t')}{\lambda_z^2(t')} - \frac{b_z(0)}{\lambda_z^2(0)} \right] \\ &+ \frac{1}{4} \int_0^{t'} \int (z_{xx}^2 + \nu z_x^2)(t'') \phi \left(\frac{5}{4} (x' - \nu t'' + y_0) \right) dx' dt'' \\ &\lesssim (\theta_z)^{\frac{1}{8}} + \int [\tilde{z}_x^2(t_0) + \tilde{z}^2(t_0)] \phi(x' + y_0) dx' + \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10} y_0}. \end{aligned} \quad (\text{A.8})$$

Undoing the transformation (A.1) and applying Lemma A.1 yields Lemma 6.2.

Proof of Lemma A.1. The proof is closely related to the argument in [24], [17].

We define for $y_0 > 1$ and $0 < \nu < \frac{1}{10}$ the following localized mass and energy quantities:

$$\begin{aligned} M_0(t') &= \int z^2(t', x') \phi(x' - \nu t' + y_0) dx', \\ E_0(t') &= \frac{1}{2} \int \left(z_x^2 - \frac{1}{3} z^6 \right) (t', x') \phi \left(\frac{5}{4} (x' - \nu t' + y_0) \right) dx'. \end{aligned}$$

step 1 Monotonicity in L^2 for z .

We claim :

$$M_0(t') - M_0(0) + \frac{1}{4} \int_0^{t'} \int (z_x^2 + \nu z^2)(t'') \phi'(x' - \nu t'' + y_0) dx' dt'' \lesssim \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10} y_0}. \quad (\text{A.9})$$

Indeed, we use formula (2.49) and (6.7) to estimate:

$$\frac{d}{dt'} M_0(t') \leq \int \left(-3z_x^2 - \frac{24}{25} \nu z^2 + \frac{5}{3} z^6 \right) \phi'(x' - \nu t' + y_0).$$

We claim that the nonlinear term¹⁶ is controllable up to an exponentially small term after integration in time. Indeed, first recall from Lemma 6 in [24] and (6.7) that for all $v \in H^1$, $a > 0$, $b \in \mathbb{R}$,

$$\begin{aligned} \|v^2(\phi')^{\frac{1}{2}}\|_{L^\infty(|x-b|>a)}^2 &\lesssim \|v\|_{L^2(|x-b|>a)}^2 \left(\int v_x^2 \phi' + \int v^2 \frac{(\phi'')^2}{\phi'} \right) \\ &\lesssim \|v\|_{L^2(|x-b|>a)}^2 \left(\int v_x^2 \phi' + \nu \int v^2 \phi' \right). \end{aligned} \quad (\text{A.10})$$

Fix $a_0 \gg 1$ such that

$$\left(\int_{2|y|>a_0} Q^2 \right)^2 \leq \delta_0.$$

On the one hand, by (A.10),

$$\begin{aligned} &\int_{|x'-x_z(t')|>a_0} z^6 \phi'(x' - \nu t' + y_0) \\ &\leq \|z\|_{L^2(|x'-x_z(t')|>a_0)}^2 \|z^2(\phi'(x' - \nu t' + y_0))^{\frac{1}{2}}\|_{L^\infty(|x'-x_z(t')|>a_0)}^2 \\ &\lesssim \|z\|_{L^2(|x'-x_z(t')|>a_0)}^4 \int (z_x^2 + \nu z^2) \phi'(x' - \nu t' + y_0). \end{aligned}$$

Since

$$\|z\|_{L^2(|x'-x_z(t')|>a_0)}^2 \lesssim \int_{\lambda_z(t')|y|>a_0} Q_b^2(y) dy + \int \varepsilon_z^2 \lesssim \delta_0 + \delta(\alpha),$$

we obtain, for δ_0 small enough and α small enough,

$$\begin{aligned} \int_{|x'-x_z(t')|>a_0} z^6 \phi'(x' - \nu t' + y_0) &\lesssim (\delta_0 + \delta(\alpha)) \int (z_x^2 + \nu z^2) \phi'(x' - \nu t' + y_0) \\ &\leq \frac{1}{4} \int (z_x^2 + \nu z^2) \phi'(x' - \nu t' + y_0). \end{aligned}$$

¹⁶which has the wrong sign

On the other hand, the modulation equation (2.29) and the upper bound on scaling (A.4) ensure:

$$(x_z)_t = \frac{1}{\lambda_z^2} \frac{(x_z)_s}{\lambda_z} \geq \frac{1 + \delta(\alpha_0)}{\lambda_z^2} \geq 0.2 \quad (\text{A.11})$$

and thus in particular:

$$x_z(t') \geq x_z(0) + 0.2t' \geq 0.1t' + \nu t', \quad (\text{A.12})$$

We then estimate from Sobolev:

$$\|z\|_{L^6}^6 \lesssim \|z\|_{H^1}^2 \|z\|_{L^2}^4 \lesssim \frac{1}{\lambda_z^2} \lesssim (x_z)_t(t'),$$

and obtain: $\forall y_0 > a_0$,

$$\begin{aligned} \int_{|x' - x_z(t')| < a_0} z^6 \phi'(x' - \nu t' + y_0) &\lesssim (x_z)_t(t') \|\phi'(x' - \nu t' + y_0)\|_{L^\infty(|x' - x_z(t')| < a_0)} \\ &\lesssim (x_z)_t(t') e^{-\frac{\sqrt{\nu}}{10}(x_z(t') - a_0 - \nu t' + y_0)} \lesssim (x_z)_t(t') e^{-\frac{\sqrt{\nu}}{100}x_z(t') - \frac{\sqrt{\nu}}{10}y_0}. \end{aligned}$$

In conclusion, we have the L^2 monotonicity formula: for all $t' \in [0, t_0]$,

$$\frac{d}{dt'} M_0(t') + \frac{1}{4} \int (z_x^2 + \nu z^2)(t') \phi'(x' - \nu t' + y_0) dx' \lesssim (x_z)_t(t') e^{-\frac{\sqrt{\nu}}{100}x_z(t')} e^{-\frac{\sqrt{\nu}}{10}y_0},$$

and by integration between 0 and t' using $x_z(0) = 0$: $\forall t' \in [0, T_z]$,

$$M_0(t') + \frac{1}{4} \int_0^{t'} \int (z_x^2 + \nu z^2)(t'') \phi'(x' - \nu t' + y_0) dx' dt'' \leq M_0(0) + \frac{C}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10}y_0}.$$

step 2 Monotonicity in L^2 for \tilde{z} . Proof of (A.7).

We now rewrite the monotonicity (A.9) using the decomposition (A.2). We compute:

$$\begin{aligned} M_0(t') &= \int z^2(t', x') \phi(x' - \nu t' + y_0) dx' \\ &= \int (Q_{b_z(t')}(y) + \varepsilon_z(y, t'))^2 \phi(\lambda_z(t')y + x_z(t') - \nu t' + y_0) dy dt' \\ &= \int Q_{b_z(t')}^2 \phi(\lambda_z(t')y + x_z(t') - \nu t' + y_0) \\ &\quad + \int \tilde{z}^2(t', x') \phi(x' - \nu t' + y_0) dx' \\ &\quad + 2 \int Q_{b_z(t')} \varepsilon_z(t') \phi(\lambda_z(t')y + x_z(t') - \nu t' + y_0) dy dt' \end{aligned}$$

We estimate using the lower bound (A.12):

$$\begin{aligned} &\int Q_{b_z(t')}^2 \phi(\lambda_z(t')y + x_z(t') - \nu t' + y_0) \\ &= \int Q^2 + 2b_z(t')(P, Q) + 2b_z(t') \int \chi_{b_x(t')} P \phi(\lambda_z(t')y + x_z(t') - \nu t' + y_0) \\ &= \int Q^2 + 2b_z(t')(P, Q) + O(e^{-\frac{\sqrt{\nu}}{10}y_0}) + O(b_z^{2-\gamma}(t')) \end{aligned}$$

where we used $b^2 \int P^2 \chi_b^2 = O(b^{2-\gamma})$. Now by Hölder:

$$\begin{aligned} & 2b_z(t') \left| \int \varepsilon_z(t') \chi_{b_z} P \phi(\lambda_z(t') y + x_z(t') - \nu t' + y_0) \right| \\ & \lesssim (b_z(t'))^{\frac{1-\gamma}{2}} \int \varepsilon_z^2(t') \phi(\lambda_z(t') y + x_z(t') - \nu t' + y_0) + (b_z(t'))^{\frac{3+\gamma}{2}} \int P^2 \chi_{b_z}^2 \\ & \leq (b_z(t'))^{\frac{1-\gamma}{2}} \int \tilde{z}^2(t', x') \phi(x' - \nu t' + y_0) dx' + (b_z(t'))^{\frac{3-\gamma}{2}}. \end{aligned} \quad (\text{A.13})$$

We now inject these estimates into (A.9) and use from (A.4) and the definition of θ_z :

$$|b_z(t')| \lesssim \theta_z, \quad (\text{A.14})$$

and thus derive from the initialization (A.3) the bound (note $\gamma = \frac{3}{4}$): $\forall t' \in [0, T_z]$,

$$\begin{aligned} & \int \tilde{z}^2(t', x') \phi(x' - \nu t' + y_0) dx + \int_0^{t'} \int (z_x^2 + \nu z^2)(t'') \phi'(x' - \nu t'' + y_0) dx' dt'' \\ & \lesssim \theta_z^{\frac{9}{8}} + \int \tilde{z}^2(0, x') \phi(x' + y_0) dx' + \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10} y_0}. \end{aligned} \quad (\text{A.15})$$

Reinjecting this bound into (A.13) and (A.9), keeping track of the b_z powers now yields (A.7).

step 3 Energy monotonicity for z .

We claim the energy monotonicity:

$$\begin{aligned} & E_0(t') - E_0(0) + \frac{1}{4} \int_0^{t'} \int (z_{xx}^2 + \nu z_x^2)(t'') \phi \left(\frac{5}{4}(x' - \nu t'' + y_0) \right) dx' dt'' \\ & \lesssim \left(\theta_z^{\frac{9}{8}} + \int \tilde{z}^2(t_0) \phi(x' + y_0) dx' + \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10} y_0} \right)^{\frac{5}{4}}. \end{aligned} \quad (\text{A.16})$$

Indeed, we estimate from formula (2.50) and (6.7):

$$\frac{d}{dt'} E_0(t') \quad (\text{A.17})$$

$$\begin{aligned} & = -\frac{5}{4} \int \left((z_{xx} + z^5)^2 + 2z_{xx}^2 - 10z^4 z_x^2 + \nu(z_x^2 - \frac{1}{3}z^6) \right) \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) \\ & + \left(\frac{5}{4} \right)^3 \int z_x^2 \phi''' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) \\ & \leq -\frac{5}{4} \int \left(2z_{xx}^2 + \frac{\nu}{2} z_x^2 - \frac{\nu}{3} z^6 - 10z^4 z_x^2 \right) \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right). \end{aligned} \quad (\text{A.18})$$

We need to treat the non linear terms. We claim:

$$\begin{aligned} & \int_0^{T_z} \int z^4 z_x^2 \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) dx' dt' \\ & \lesssim \delta_0 \int_0^{T_z} \int (z_{xx}^2 + \nu z_x^2) \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) dx' dt' + \frac{1}{\sqrt{\nu}} e^{-\frac{5}{4} \frac{\sqrt{\nu}}{10} y_0} \\ & + \int_0^{T_z} \int z^6 \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) dx' dt', \end{aligned} \quad (\text{A.19})$$

for some small enough $\delta_0 > 0$, and

$$\begin{aligned} & \int_0^{T_z} \int z^6(t') \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) dt' \\ & \lesssim \left(\theta_z^{\frac{9}{8}} + \int \tilde{z}^2(t_0) \phi(x' + y_0) dx' + \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10} y_0} \right)^{\frac{5}{4}}. \end{aligned} \quad (\text{A.20})$$

Integrating (A.18) in time and injecting (A.19), (A.20) yields (A.16).

Proof of (A.19): For $a_1 > 0$ large enough, we have¹⁷

$$\frac{1}{\lambda_z^2(t')} \int_{|x| > a_1} (Q')^2 \left(\frac{x}{\lambda_z(t')} \right) dx \lesssim \frac{1}{\lambda_z^2(t')} e^{-\frac{2a_1}{\lambda_z(t')}} \lesssim \frac{1}{a_1^2} \leq \delta_0,$$

then:

$$\begin{aligned} \int_{|x-x_z(t')| > a_1} z_x^2 & \lesssim \int_{|x-x_z(t')| > a_1} \tilde{z}_x^2(x) + \frac{1}{\lambda_z^2(t')} \int_{|x| > a_1} (Q')^2 \left(\frac{x}{\lambda_z(t')} \right) \\ & \lesssim \delta_0 \end{aligned} \quad (\text{A.21})$$

where we used the smallness in the H^1 bound (A.5).

We now write

$$\begin{aligned} & \int_{|x-x_z(t)| > a_1} z^4 z_x^2 \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) \\ & \lesssim \int_{|x-x_z(t)| > a_1} (z^2 z_x^4 + z^6) \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right), \end{aligned}$$

and need only treat the first term according to the expected bound (A.19). We estimate the outer integral by using the localized Gagliardo-Nirenberg inequality (A.10) and the outer smallness by (A.21):

$$\begin{aligned} & \int_{|x-x_z(t)| > a_1} z^2 z_x^4 \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) \\ & \lesssim \|z_x^2 (\phi'(\frac{5}{4}(x' - \nu t' + y_0)))^{\frac{1}{2}}\|_{L^\infty(|x-x_z(t')| > a_1)}^2 \|z\|_{L^2}^2 \\ & \lesssim \|z_x\|_{L^2(|x-x_z(t')| > a_1)}^2 \int (z_{xx}^2 + \nu z_x^2) \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) \\ & \lesssim \delta_0 \int (z_{xx}^2 + \nu z_x^2) \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right). \end{aligned}$$

The inner integral is estimated from Sobolev

$$\int z^4 z_x^2 \lesssim \|z\|_{L^\infty}^4 \|z_x\|_{L^2}^2 \lesssim \|z\|_{L^2}^2 \|z_x\|_{L^2}^4 \lesssim \frac{1}{\lambda_z^4},$$

and hence using the structure of ϕ and (A.12):

$$\begin{aligned} & \int_{|x-x_z(t)| < a_1} z^4 z_x^2 \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) \\ & \lesssim \frac{1}{\lambda_z^4(t')} \|\phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right)\|_{L^\infty(|x-x_z(t')| < a_1)} \\ & \lesssim \frac{1}{\lambda_z^4(t')} e^{-\frac{\sqrt{\nu}}{100} x_z(t') - \frac{5}{4} \frac{\sqrt{\nu}}{10} y_0}. \end{aligned}$$

¹⁷using $x^2 e^{-x} \lesssim 1$ for $x \geq 0$

We now claim

$$\frac{1}{c_0 \lambda_z^2(t')} e^{-c_0 x_z(t')} + \int_0^{T_z} \frac{1}{\lambda_z^4(t')} e^{-c_0 x_z(t')} dt' \lesssim \frac{1}{c_0} \quad (\text{A.22})$$

with $c_0 = C\sqrt{\nu}$, which completes the proof of (A.19).

Indeed, first observe from the definition of θ_z and the rough modulation equation (2.29):

$$|(\lambda_z)_t| = \left| \frac{1}{\lambda_z^2} \frac{-(\lambda_z)_s}{\lambda_z} \right| \lesssim \frac{1}{\lambda_z^2} (|b_z| + \sqrt{\theta_z} \lambda_z) \lesssim \frac{\sqrt{\theta_z}}{\lambda_z},$$

and thus from (A.11) and an integration by parts in time:

$$\begin{aligned} & \int_0^{t'} \frac{1}{\lambda_z^4} e^{-c_0 x_z} d\tau \lesssim \int_0^{t'} \frac{(x_z)_t}{\lambda_z^2} e^{-c_0 x_z} d\tau \\ &= \left[\frac{-1}{c_0 \lambda_z^2} e^{-c_0 x_z} \right]_0^{t'} - \frac{1}{c_0} \int_0^{t'} \frac{2(\lambda_z)_t}{\lambda_z^3} e^{-c_0 x_z} d\tau \\ &\leq \frac{1}{c_0} \left[1 - \frac{1}{\lambda_z^2(t')} e^{-c_0 x_z(t')} \right] + \frac{2\sqrt{\theta_z}}{c_0} \int_0^{t'} \frac{1}{\lambda_z^4} e^{-c_0 x_z} d\tau, \end{aligned}$$

and (A.22) now follows from the a priori smallness (A.6), (6.6).

Proof of (A.20): Since $\phi'(\frac{5}{4}x) \lesssim (\phi')^{\frac{5}{4}}(x)$, (A.10) yields:

$$\begin{aligned} & \int z^6 \phi' \left(\frac{5}{4}(x' - \nu t' + y_0) \right) \leq \|z^2(\phi')^{\frac{1}{2}}(x' - \nu t' + y_0)\|_{L^\infty}^2 \int z^2(\phi')^{\frac{1}{4}}(x' - \nu t' + y_0) \\ &\lesssim \left(\int z^2 \right)^{\frac{7}{4}} \left(\int z^2 \phi'(x' - \nu t' + y_0) \right)^{\frac{1}{4}} \int (z_x^2 + \nu z^2) \phi'(x' - \nu t' + y_0) \\ &\lesssim \left(\int z^2 \phi'(x' - \nu t' + y_0) \right)^{\frac{1}{4}} \int (z_x^2 + \nu z^2) \phi'(x' - \nu t' + y_0). \end{aligned}$$

We now estimate:

$$\begin{aligned} & \int z^2 \phi'(x' - \nu t' + y_0) \\ &\lesssim \int \tilde{z}^2 \phi'(x' - \nu t' + y_0) + \int Q_{b_z}^2(y) \phi'(\lambda_z(t')y + x_z(t) - \nu t' + y_0). \end{aligned}$$

On the one hand by (A.15) and $\phi' \lesssim \phi$:

$$\int \tilde{z}^2 \phi'(x' - \nu t' + y_0) \lesssim \theta_z^{\frac{9}{8}} + \int \tilde{z}^2(0) \phi(x' + y_0) dx' + \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10} y_0}.$$

On the other hand, from the space decoupling (A.12):

$$\begin{aligned} & \int Q_b^2(y) \phi'(\lambda_z(t')y + x_z(t) - \nu t' + y_0) \\ &\lesssim |b|^{2-\gamma}(t') + \int Q^2(y) \phi'(\lambda_z(t')y + x_z(t) - \nu t' + y_0) \\ &\lesssim \theta_z^{\frac{5}{4}} + \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10} y_0}. \end{aligned}$$

The space-time estimate (A.20) now follows from (A.15).

step 4 Energy monotonicity for \tilde{z} . Proof of (A.8).

We now rewrite the monotonicity (A.16) using the decomposition (A.2). We compute:

$$\begin{aligned} & 2\lambda_z^2(t')E_0(t') \\ &= \int \left[(Q_{b_z} + \varepsilon_z)_y^2 - \frac{1}{3}(Q_{b_z} + \varepsilon_z)^6 \right] (t', y) \phi \left(\frac{5}{4}(\lambda_z(t')y + x_z(t') - \nu t' + y_0) \right) dy \end{aligned}$$

and develop this expression. The contribution of the Q_b term is estimated using $E(Q) = 0$ and the separation in space (A.12) which implies:

$$\begin{aligned} & \int [(Q_b)_y^2 + Q_b^6] \left[1 - \phi \left(\frac{5}{4}(\lambda_z(t')y + x_z(t') - \nu t' + y_0) \right) \right] dy \\ & \lesssim |b_z|^{1+\gamma} + \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10}(x_z(t') + y_0)}. \end{aligned}$$

The cross terms are treated using the orthogonality condition (2.20) and we obtain similarly like for the proof of (2.28):

$$\begin{aligned} & 2\lambda_z^2(t')E_0(t') \tag{A.23} \\ &= -2b_z(t')(P, Q) + \int \left[(\varepsilon_z)_y^2 - \frac{1}{3}\varepsilon_z^6 \right] (t', y) \phi \left(\frac{5}{4}(\lambda_z(t')y + x_z(t') - \nu t' + y_0) \right) dy \\ &+ O \left[\frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10}(x_z(t') + y_0)} + |b_z(t')|^{1+\gamma} + |b_z(t')|^{1-\gamma} \left(\int (\varepsilon_z)_y^2 + \int \varepsilon_z^2 e^{-|y|} \right) \right]. \end{aligned}$$

We now divide by $\lambda_z(t')$. We estimate from (A.4):

$$\frac{1}{\lambda_z^2(t')} \left[|b_z(t')|^{1+\gamma} + |b_z(t')|^{1-\gamma} \left(\int (\varepsilon_z)_y^2 + \int \varepsilon_z^2 e^{-|y|} \right) \right] \lesssim (\theta_z)^{\frac{1}{8}}$$

and conclude using (A.22), (A.23):

$$\begin{aligned} 2E_0(t') &= -\frac{2b_z(t')}{\lambda_z^2(t')}(P, Q) + \int \left[(\tilde{z})_x^2 - \frac{1}{3}\tilde{z}^6 \right] \phi \left(\frac{5}{4}(x' - \nu t' + y_0) \right) dx' \\ &+ O \left((\theta_z)^{\frac{1}{8}} + \frac{1}{\sqrt{\nu}} e^{-\frac{\sqrt{\nu}}{10}y_0} \right). \end{aligned}$$

which together with the monotonicity (A.16) and L^2 smallness of \tilde{z} yields (A.8). \square

A.2. Proof of Lemma 3.4. The proof of Lemma 3.4 is based on coercivity properties of the viriel quadratic form under suitable repulsivity properties. We recall this property in the following lemma.

Lemma A.2 ([15], Proposition 4). *There exists $\mu > 0$ such that, for all $v \in H^1(\mathbb{R})$,*

$$\begin{aligned} & 3 \int v_y^2 + \int v^2 - 5 \int Q^4 v^2 + 20 \int y Q' Q^3 v^2 \\ & \geq \mu \int v_y^2 + v^2 - \frac{1}{\mu} \left(\int v y \Lambda Q \right)^2 - \frac{1}{\mu} \left(\int v Q \right)^2 \end{aligned}$$

We now turn to the proof of Lemma 3.4 which is a simple consequence of Lemma A.2 using a standard localization argument (see for example the proof of Proposition 9 in [17]). Indeed, let ζ be a smooth function such that

$$\zeta(y) = 0 \text{ for } |y| > \frac{1}{4}; \quad \zeta(y) = 1 \text{ for } |y| < \frac{1}{8}; \quad 0 \leq \zeta \leq 1 \text{ on } \mathbb{R}.$$

Set

$$\tilde{\varepsilon}(y) = \varepsilon(y)\zeta_B(y) \quad \text{where} \quad \zeta_B(y) = \zeta\left(\frac{y}{B}\right).$$

Lemma A.2 applied to $\tilde{\varepsilon}$ gives

$$\begin{aligned} & (3 - \mu) \int \tilde{\varepsilon}_y^2 + (1 - \mu) \int \tilde{\varepsilon}^2 - 5 \int Q^4 \tilde{\varepsilon}^2 + 20 \int y Q' Q^3 \tilde{\varepsilon}^2 \\ & \geq -\frac{1}{\mu} \left(\int \tilde{\varepsilon} y \Lambda Q \right)^2 - \frac{1}{\mu} \left(\int \tilde{\varepsilon} Q \right)^2 \end{aligned} \quad (\text{A.24})$$

On the one hand,

$$\begin{aligned} \int \tilde{\varepsilon}_y^2 &= \int \varepsilon_y^2 \zeta_B^2 + \int \varepsilon^2 (\zeta_B')^2 - \frac{1}{2} \int \varepsilon^2 (\zeta_B^2)'' \leq \int_{|y| < \frac{B}{4}} \varepsilon_y^2 + \frac{C}{B^2} \int_{|y| < \frac{B}{4}} \varepsilon^2, \\ \int \tilde{\varepsilon}^2 &= \int \varepsilon^2 \zeta_B^2 \leq \int_{|y| < \frac{B}{4}} \varepsilon^2, \end{aligned}$$

and by $yQ' < 0$ and then by the exponential decay of Q and Q'

$$\begin{aligned} & -5 \int Q^4 \tilde{\varepsilon}^2 + 20 \int y Q' Q^3 \tilde{\varepsilon}^2 \leq -5 \int_{|y| < \frac{B}{4}} Q^4 \tilde{\varepsilon}^2 + 20 \int_{|y| < \frac{B}{4}} y Q' Q^3 \tilde{\varepsilon}^2 \\ & \leq -5 \int_{|y| < \frac{B}{2}} Q^4 \tilde{\varepsilon}^2 + 20 \int_{|y| < \frac{B}{2}} y Q' Q^3 \tilde{\varepsilon}^2 + C e^{-\frac{B}{16}} \int_{\frac{B}{4} < |y| < \frac{B}{2}} \varepsilon^2. \end{aligned}$$

Thus, for B large,

$$\begin{aligned} & (3 - \mu) \int \tilde{\varepsilon}_y^2 + (1 - \mu) \int \tilde{\varepsilon}^2 - 5 \int Q^4 \tilde{\varepsilon}^2 + 20 \int y Q' Q^3 \tilde{\varepsilon}^2 \leq (3 - \mu) \int_{|y| < \frac{B}{4}} \varepsilon_y^2 \\ & + (1 - \mu) \int_{|y| < \frac{B}{4}} \varepsilon^2 - 5 \int_{|y| < \frac{B}{2}} Q^4 \tilde{\varepsilon}^2 + 20 \int_{|y| < \frac{B}{2}} y Q' Q^3 \tilde{\varepsilon}^2 + \frac{C}{B^2} \int_{|y| < \frac{B}{4}} \varepsilon^2 \\ & \leq (3 - \mu) \int_{|y| < \frac{B}{2}} \varepsilon_y^2 + \left(1 - \frac{\mu}{2}\right) \int_{|y| < \frac{B}{2}} \varepsilon^2 - 5 \int_{|y| < \frac{B}{2}} Q^4 \tilde{\varepsilon}^2 + 20 \int_{|y| < \frac{B}{2}} y Q' Q^3 \tilde{\varepsilon}^2 \end{aligned}$$

On the other hand, by (2.20),

$$\left| \int \tilde{\varepsilon} y \Lambda Q \right| = \left| \int \varepsilon \zeta_B y \Lambda Q \right| = \left| \int \varepsilon (1 - \zeta_B) y \Lambda Q \right| \lesssim e^{-\frac{B}{16}} \left(\int \varepsilon^2 e^{-\frac{|y|}{2}} \right)^{\frac{1}{2}}$$

and similarly for $\int \tilde{\varepsilon} Q$. Inserted in (A.24), these estimates finish the proof of Lemma A.2.

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